Wave Polarization

Generally, a $z$-propagating plane wave has both field components in $x$ and $y$ components.

$$\vec{E}(z) = \hat{x}\vec{E}_x(z) + \hat{y}\vec{E}_y(z)$$
$$= \hat{x}E_{x0}e^{-jkz} + \hat{y}E_{y0}e^{-jkz}$$

Relative phase difference between $E_{x0}$ and $E_{y0}$ (complex numbers) determine the wave polarization.

Define $E_{x0} = a_x$ and $E_{y0} = a_y e^{i\delta}$ \hspace{1cm} $\delta$: Relative phase

$$\vec{E}(z) = (\hat{x}a_x + \hat{y}a_y e^{i\delta})e^{-jkz}$$

In the time domain

$$\overline{E}(z, t) = \text{Re}\left[\vec{E}e^{j\omega t}\right]$$
$$= \hat{x}a_x \cos(\omega t - kz) + \hat{y}a_y \cos(\omega t - kz + \sigma)$$

Total magnitude of the $E$ field

$$|\overline{E}(z, t)| = \left[|E_x^2(z, t) + E_y^2(z, t)|\right]^{1/2}$$
$$= [a_x^2 \cos^2(\omega t - kz) + a_y^2 \cos^2(\omega t - kz + \delta)]^{1/2}$$

Inclination angle of $E$ in $x$-$y$ plane

$$\psi(z, t) = \tan^{-1}\left(\frac{E_y(z, t)}{E_x(z, t)}\right)$$
**Linear Polarization**

*E* field direction traces a straight line

Condition: $E_x(z, t)$ and $E_y(z, t)$ are either

In-phase ($\delta = 0$) or out-of-phase ($\delta = \pi$)

\[
E(0,t) = (\hat{x}a_x + \hat{y}a_y)\cos \omega t \quad \text{(in-phase)}
\]
\[
E(0,t) = (\hat{x}a_x - \hat{y}a_y)\cos \omega t \quad \text{(out-of-phase)}
\]

The modulus of $E(0,t)$ varies as $\cos \omega t$ in time.

\[
|E(0,t)| = [a_x^2 + a_y^2]^{1/2} \cos \omega t
\]

The inclination angle is given by

\[
\psi = \tan^{-1}\left(\frac{a_y}{a_x}\right) \quad \text{(in-phase)}
\]
\[
= \tan^{-1}\left(\frac{-a_y}{a_x}\right) \quad \text{(out-of-phase)}
\]

$\psi$ is independent of both $z$ and $t$.

$E(z,t)$ maintains a direction along the line making an angle $\psi$ with the $x$ axis.

**Special case:** $a_y = 0$, $\psi = 0^\circ$ or $180^\circ$, $x$-polarized

$a_x = 0$, $\psi = 90^\circ$ or $-90^\circ$, $y$-polarized
Circular Polarization

E field vector traces a circle in the $x-y$ plane

Condition: $\delta = \pm \pi/2$, \hspace{1em} \left| E_{x0} \right| = \left| E_{y0} \right| \hspace{1em} (a_x = a_y)$

Left-hand circular if $\delta = \pi/2$
Right-hand circular if $\delta = -\pi/2$

(See the thumb of left/right hand in $z$ and rotation along fingers)

Left-Hand Circular (LHC) Polarization

$a_x = a_y = a$, $\delta = \pi/2$,

$$\vec{E}(z) = (\hat{x}a + \hat{y}ae^{i\pi/2})e^{-jkz} = a(\hat{x} + j\hat{y})e^{-jkz}$$

Time domain

$$\vec{E}(z,t) = \text{Re}\left[\vec{E}(z)e^{j\omega t}\right]$$

$$= \hat{x}a \cos(\omega t - kz) + \hat{y}a \cos(\omega t - kz + \pi/2)$$

$$= \hat{x}a \cos(\omega t - kz) + \hat{y}a \sin(\omega t - kz)$$

Modulus:

$$\left| \vec{E}(z,t) \right| = \left[ E_x^2(z,t) + E_y^2(z,t) \right]^{1/2}$$

$$= [a^2 \cos^2(\omega t - kz) + a^2 \sin^2(\omega t - kz)]^{1/2} = a$$

Inclination angle

$$\psi(z,t) = \tan^{-1} \frac{E_y(z,t)}{E_x(z,t)} = \tan^{-1} \frac{-a \sin(\omega t - kz)}{a \cos(\omega t - kz)}$$

$$= -(\omega t - kz)$$
(a) LHC polarization

(b) RHC polarization
Right Hand Circular (RHC) Polarization

For $a_x = a_y = a$ and $\delta = -\pi/2$,

\[ ||\vec{E}(z,t)|| = a \text{ and } \psi = (\omega t - kz) \]

The trace of $E$ field is a function of time.
*The traces in $z$ direction rotate in opposite way to that in time for both LHC and RHC.*

Note $\psi = -\omega t + kz$ \hspace{1cm} (LHC)
$\psi = +\omega t - kz$ \hspace{1cm} (RHC)

The sign of $t$ dependent term and $z$ dependent term are always different so that we have
Elliptical Polarization

More general case: \( a_x \neq 0, a_y \neq 0 \) and \( \delta \neq 0, a_x \neq a_y \)
The tip of E vector traces on an ellipse in the \( x-y \) plane inclined by \( \gamma \) (major axis and \( x \) axis).

Ellipticity angle \( \chi \)
\[
\tan \chi = \pm \frac{a_y}{a_x} = \pm \frac{1}{R}
\]

\( R \): Axial ratio
- \( R = 1 \) Circular
- \( R = \infty \) Linear

+ Left handed
- Right handed
\(-\pi/4 \leq \chi \leq \pi/4\)

Relationship between \( \gamma \) and \( \chi \)
\[
\tan 2\gamma = (\tan 2\psi_0) \cos \delta \quad (-\pi/2 \leq \gamma \leq \pi/2)
\]
\[
\tan 2\chi = (\sin 2\psi_0) \sin \delta \quad (-\pi/4 \leq \chi \leq \pi/4)
\]

\( \psi_0 \) is defined by \( \tan \psi_0 = a_y/a_x \) (0 \( \leq \psi_0 \leq \pi/2 \))

Select sign of \( \gamma \)
- \( \gamma > 0 \) if \( \cos \delta > 0 \)
- \( \gamma < 0 \) if \( \cos \delta < 0 \)