**FREQUENCY RESPONSE**

Devices known as filters are used to separate one band of frequencies from another band of frequencies.

- Crossover Network

\[ V_s(t) \]

\[ L \quad 8 \text{ohm} \]

\[ \text{Tweeter} \quad 8 \text{ohm} \]

\[ 8 \text{ohm} \]

\[ \text{Wooper} \]

- The inductor presents high impedance at high frequencies.
- The capacitor presents high impedance for low frequencies.

- The crossover frequency

\[
\left| j \omega L \right| = \left| \frac{1}{j \omega C} \right| \quad \omega_r = \sqrt{\frac{1}{LC}}
\]

- If \( \omega_r = \frac{2 \pi}{1000 \text{ Hz}} = \frac{6283}{2 \pi} \text{ rad/sec} \)

\[
L \times C = \frac{1}{\omega_r^2} = 2.53 \times 10^{-8}
\]

- The impedance is

\[
Z(w) = \frac{(8 + j \omega L)(8 - j \omega C)}{(8 + j \omega L)(8 - j \omega C)}
\]
• For low $\omega$, $\omega L \rightarrow 0$ the load is $8 \Omega$.
  For high $\omega$, $\frac{1}{\omega C} \rightarrow 0$ the load is $8 \Omega$.
• For resonance, $\omega L = \frac{1}{\omega C}$.
  
\[ Z(\omega) = 4 + \frac{L}{16 C} \]

• For an impedance match at $\omega R$,
  
\[ Z(\omega_R) = 4 + \frac{L}{16 C} = 8 \]
  
\[ L = 64 C \]

  Can obtain $C = 19.9 \mu F$.
  
\[ L = 1.27 \mu H \]

• Thus crossover impedance is constant at all frequencies.

• Many applications of filtered exist in communications and sound reproduction.
FREQUENCY RESPONSE

- FILTER NETWORKS

- DEFINE TRANSFER FUNCTION

\[ H(f) = \frac{V_{\text{out}}}{V_{\text{in}}} \]

A complex function \( A(f) e^{j\phi(f)} \)

- AMPLITUDE AND PHASE PROPERTIES

\[ |H(f)| \]

\[ \angle H(f) \]

- \( V_{\text{in}} = 2 \cos(2\pi \cdot 1000t + 40^\circ) \)
- \( V_{\text{in}} = 2 \angle 40 \)
- \( H(1000) = 3 \angle 30 = \frac{V_{\text{out}}}{V_{\text{in}}} \)
- \( V_{\text{out}} = 3 \angle 30 \cdot 2 \angle 40 = 6 \angle 70^\circ \)

Thus

\[ V_{\text{out}}(t) = 6 \cos(2\pi \cdot 1000t + 70^\circ) \]
**TWO SIGNALS**

\[ V_{in} = 2 \cos(2\pi \cdot 1000t) + \cos(2\pi \cdot 2000t - 70^\circ) \]

\[ H(1000) = 3 \angle 30^\circ \quad H(2000) = 2 \angle 60^\circ \]

\[ \frac{V_{out}(1000)}{V_{in}(1000)} = \frac{V_{out}(2000)}{V_{in}(2000)} \]

\[ V_{out}(1000) = 2 \angle 0 \cdot 3 \angle 30^\circ \quad V_{out}(2000) = 1 \angle -70 \cdot 2 \angle 60^\circ \]

**RECONSTITUTING TIME FUNCTIONS**

\[ V_{out}(t) = 6 \cos(2\pi \cdot 1000t + 30^\circ) \]

\[ + 2 \cos(2\pi \cdot 2000t - 10^\circ) \]

**This example is only for 2 frequencies**

A TYPICAL FILTER HAS MANY FREQUENCIES

WE MUST CALCULATE THE EFFECT OF THE FILTER ON EACH FREQUENCY PRESENT IN THE SIGNAL AND ADD THEM.

**MEASURING THE TRANSFER FUNCTION**

\[ V_{in}(t) \]

\[ V_{out}(t) \]

**CHANGE INPUT FREQUENCY AND MEASURE**

\[ V_{out} / V_{in} \] (both amplitude and phase)

**A LOW PASS R-C NETWORK**
\[ Z_L = R \]
\[ Z_C = \frac{1}{j2\pi fC} \]
\[ Z_{series} = R + \frac{1}{j2\pi fC} \]

\[ \Pi = \frac{V_{in}}{Z} = \frac{V_{in}}{R + \frac{1}{j2\pi fC}} \]

\[ V_{out} = Z_C \Pi = \frac{1}{j2\pi fC} \times \frac{V_{in}}{R + \frac{1}{j2\pi fC}} \]

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{1}{1 + j2\pi fRC} \]

@ \( f = 0 \) \( H(f) = 1 \)

@ \( f = \frac{1}{2\pi RC} \) \( H(f) = \frac{1}{1 + j} = \frac{1}{\sqrt{2} - j\pi fC} \)

THE AMPLITUDE IS DOWN BY \( \frac{1}{\sqrt{2}} \)
THE POWER IS DOWN BY \( \frac{1}{2} \) \( (3 dB) \)

**GENERALIZED VARIABLES**

- LET \( f_B = \frac{1}{2\pi RC} \)

- \( H(f) = \frac{1}{1 + j \frac{f}{f_B}} \)

- \( |H(f)| = \left( \frac{1}{\sqrt{1 + (f/f_B)^2}} \right) \]

\[ \angle H(f) = -\pi \tan^{-1} \left( \frac{f}{f_B} \right) \]

\[ 0 \rightarrow 0.707 \rightarrow 0.5 \]

\[ 0 \rightarrow 90 \rightarrow 45 \]
Look at a specific filter

\[ V_{in}(t) = V_{in1} + V_{in2} + V_{in3} \]

\[ V_{in1} = 5 \cos 20\pi t \quad V_{in2} = 5 \cos 200\pi t \quad V_{in3} = 5 \cos 2000\pi t \]

\[ f = 10 \text{ Hz} \quad f = 100 \text{ Hz} \quad f = 1000 \text{ Hz} \]

\[ H_1(\alpha) = \frac{1}{1 + j(100\alpha)} \quad H_2(\alpha) = \frac{1}{1 + j(10^2)} \quad H_3(\alpha) = \frac{1}{1 + j(10^3)} \]

\[ H_1(10) = 0.995L5.71^\circ \quad H_2(100) = 0.707L45^\circ \quad H_3(1000) = 0.0995L84.29^\circ \]

Combining terms

\[ V_{out}(1) = 4.975L5.71^\circ \]
\[ V_{out}(2) = 3.535L-45^\circ \]
\[ V_{out}(3) = 0.4975L-84.29^\circ \]

Including time dependence

\[ V_{out}(t) = 4.975 \cos (20\pi t - 5.71^\circ) + 3.535 \cos (200\pi t - 45^\circ) + 0.4975 \cos (2000\pi t - 84.29^\circ) \]

Note

- Amplitude of second component reduced by 0.707 x (at output)
- Amplitude of third component reduced by 0.1 x (at output)
**DECIBELS**

\[ |H(f)|_{db} = 20 \log_{10} |H(f)| \]

**DEFINITION**

**EXAMPLE**

\[ |H(f)| = 100 \quad |H(f)|_{db} = 20 \log_{10} 10^2 = 40 \text{ db} \]
\[ = 20 \text{ db} \]
\[ = 6 \text{ db} \]
\[ = 3 \text{ db} \]
\[ = 0 \]
\[ = -3 \]
\[ = -6 \]

**CASCADED FILTERS**

\[ V_{in} = V_{in1}, \quad V_{out1} = V_{in2}, \quad V_{out2} = V_{out} \]

\[ H_1(f) \quad \quad H_2(f) \]

**DEF**

\[ H(f) = \frac{V_{out}}{V_{in}} = \frac{V_{out_2}}{V_{in_1}} \]

Multiply and divide by \( V_{out_1} \)

\[ H(f) = \frac{V_{out_1}}{V_{in_1}} \times \frac{V_{out_2}}{V_{out_1}} = \frac{V_{out_1}}{V_{in_2}} \]

\[ H(f) = \frac{V_{out_1}}{V_{in_1}} \times \frac{V_{out_2}}{V_{in_2}} \equiv H_1(f) \times H_2(f) \]

\[ 20 \log_{10} |H(f)| = 20 \log_{10} |H_1(f)| + 20 \log_{10} |H_2(f)| \]
LOG FREQUENCY PLOTS (USUALLY TO SHRINK SIZE OF PLOT)

- A DECADE = 10:1 RATIO OF 2 FREQUENCIES (10 kHz - 1 kHz)
- AN OCTAVE = 2:1 RATIO OF 2 FREQUENCIES (20 kHz - 10 kHz)
- NO = # OF DECADES = \( \log_{10} \frac{f_2}{f_1} \)
- NO = # OF OCTAVES = \( \frac{\log_{10}(f_2/f_1)}{\log_{10}(2)} \)

BODE PLTOS

EXAMPLE LOW PASS FILTER

\[ H(f) = \frac{1}{1 + j(f/f_B)} \quad |H(f)| = \frac{1}{\sqrt{1 + (f/f_B)^2}} \]

\[ (H(f))_{dB} = 20 \log_{10} \left| \frac{1}{\sqrt{1 + (f/f_B)^2}} \right| \]

\[ = -20 \times \frac{1}{2} \log_{10} \left[1 + \left(\frac{f}{f_B}\right)^2\right] \]

\[ |H(f)|_{dB} = -10 \log_{10} \left[1 + \left(\frac{f}{f_B}\right)^2\right] \]

FOR BODE PLOT USE \( f/f_B = \frac{1}{100} \), \( 1 \), \( 10 \), \( 100 \)

\[ |H(f)|_{dB} \] vs \( f/f_B \)

- THE SLOPE IS \( -20 \text{ dB per decade change in frequency} \)
- OR \( -20 \text{ dB per octave change in frequency} \)

HIGH FREQUENCY ASYMPTOTE

For \( f > f_B \)

\[ |H(f)|_{dB} = -20 \log_{10} \left(\frac{f}{f_B}\right) \]

For \( \frac{f}{f_B} = 10 \), \( -20 \log_{10} 10 = -20 \text{ dB} \)

\[ \frac{f}{f_B} = 100 \]
AN RC HIGH PASS FILTER

\[ V_{\text{out}} = \frac{R}{V_s} \frac{R}{R + \frac{1}{j\omega CR}} \]

\[ V_{\text{out}} = \frac{R}{R + \frac{1}{j\omega CR}} = \frac{(f/f_B)}{(f/f_B) + 1} \]

\[ |H(f)| = \frac{f/f_B}{\sqrt{1 + (f/f_B)^2}} \]

\[ \angle H(f) = 90 - \tan^{-1}(f/f_B) \]

DB - BODE PLOT

\[ |H(f)|_{dB} \]

\[ \angle H(f) \]
SERIES RESONANCE

\[ Z = j \frac{2 \pi f L}{R} + \frac{1}{j \frac{2 \pi f C}{R}} \]

- At resonance:
  \[ 2\pi f_0 L = \frac{1}{2\pi f_0 C} \]
  \[ f_0 = \frac{1}{2\pi \sqrt{L C}} \]

- Using \( f_0 \) as a parameter:
  \[ Z = R \left[ 1 + j Q_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

- Solving for \( I \):
  \[ I = \frac{V_s}{Z} = \frac{V_s}{R} \left[ 1 + j Q_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

- Solving for \( V_R \):
  \[ V_R = I R = \frac{V_s}{1 + j Q_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]

\[ \left| \frac{V_R}{V_s} \right| = \frac{1}{\sqrt{1 + Q_s^2 \left( \frac{f}{f_0} - \frac{f_0}{f} \right)^2}} \]

\[ \left| \frac{V_R}{V_s} \right| = 0.707 \]

- Half power bandwidth (HPBW): \( f_{H} - f_{L} \)

\[ Q_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) = \pm 1 \]
• \( \nu_n = 0.707 \) \( g_s \left( \frac{f_0}{f_0} - \frac{f_0}{f} \right) = \pm 1 \)

\[ f^2 = f_0^2 \pm \frac{f_0^2}{g_s} = 0 \]

\[ f = \pm \frac{f_0}{2g_s} \pm \sqrt{\left( \frac{f_0}{2g_s} \right)^2 + f_0^2} \]

• For \( g_s > 10 \) \( f = \pm \frac{f_0}{2g_s} \left( 1 \pm 2g_s \right) \)

\[ f_1 = \frac{f_0}{2g_s} (-1 + 2g_s) \quad f_2 = \frac{f_0}{2g_s} (1 + 2g_s) \]

\[ B = f_0 - f_1 = \frac{f_0}{2g_s} \left[ 1 + 2g_s - (-1 + 2g_s) \right] \]

\[ B = \frac{f_0}{g_s} \quad \text{(HPBW)} \]

\[ f_H = f_0 + B/2 \quad f_L = f_0 - B/2 \]

\[ C = 0.1592 \text{M} \]

A NUMERICAL EXAMPLE

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} = \frac{1}{2\pi \sqrt{0.1592 \times 1592 \times 10^{-6}}} = 1000 \text{ Hz} \]

\[ \gamma_s = \frac{2\pi f_0 L}{R} = \frac{2\pi \times 10^3 \times 0.1592}{100} = 10 \]

\[ B = \frac{f_0}{\gamma_s} = \frac{1000}{10} = 100 \text{ Hz} \]

\[ f_H = f_0 + B/2 = 1050 \text{ Hz} \]

\[ f_L = f_0 - B/2 = 950 \text{ Hz} \]

\[ \text{At resonance} \ f = f_0 = 1000 \text{ Hz} \]

\[ Z_L = j 2\pi f_0 L = j 2\pi \times 10^3 \times 0.1592 = j 1000 \text{ ohm} \]

\[ Z_C = -j \frac{1}{2\pi f_0 C} = -j \frac{1}{2\pi \times 10^3 \times 0.1592 \times 10^{-6}} = -j 1000 \]

\[ Z_S = R + Z_L + Z_C = 100 + 31000 - 3100 = 100 \]
\[ \text{Phasor } I = \frac{V_s}{2 \pi s} = \frac{100}{100} \angle 0^\circ = 0.01 \angle 0^\circ \]

- **Phasor Voltages**
  \[ V_L = R I = 100 \times 0.01 \angle 0^\circ = 1 \angle 0^\circ \]
  \[ V_L = Z_L I = 3 \times 1000 \times 0.01 \angle 0^\circ = 10 \angle 90^\circ \]
  \[ V_C = Z_C I = -3 \times 1000 \times 0.01 \angle 0^\circ = 10 \angle -90^\circ \]

- **Inductive Voltage Component**
  Cancels Capacitive Voltage Component

- Phasor Resultant is Real. It has the same phase as the source.

---

**Parallel Resonance**

\[ I = I_0 \angle 0^\circ \]

\[ G_p = \frac{R}{2 \pi f_0 L} \quad \text{or} \quad 2 \pi f_0 CR \]

\[ Z_p = \frac{R}{1 + j \omega_p C (\omega_p - \frac{1}{\omega_p})} \]

\[ V_{\text{out}} = I Z_p = \frac{I R}{1 + j \omega_p C (\omega_p - \frac{1}{\omega_p})} \]
\[ B = f_H - f_L = f_0 / \Phi_p \]

**Example**

\[ \Omega = 10^{-3} \angle 0^\circ \quad R = 10 \Omega \quad f_0 = 1 \times 10^6 \quad B = 100 \text{KHz} \]

\[ \Phi_p = f_0 / R = \frac{10^6}{10} = 10 \]

\[ L = \frac{R}{2\pi f_0 \Phi_p} = \frac{10}{2\pi \times 10^6 \times 10} = 159.2 \mu \text{H} \]

\[ C = \frac{\Phi_p}{2\pi f_0 L} = \frac{10}{2\pi \times 10^6 \times 10^4} = 159.2 \text{ pF} \]

**At Resonance**

\[ V_{out} = \Omega R = 10^{-3} \angle 0^\circ \times 10^4 = 10 \angle 0^\circ \]

\[ I_R = \frac{V_{out}}{R} = \frac{10 \angle 0^\circ}{10^4} = 10^{-3} \angle 0^\circ \]

\[ I_L = \frac{V_{out}}{2\pi f_0 L} = \frac{10 \angle 0^\circ}{2 \pi \times 10^3} = 10^{-2} \angle 90^\circ \]

\[ I_C = \frac{V_{out}}{2\pi f_0 C} = \frac{10 \angle 0^\circ}{2 \pi \times 10^3} = 10^{-2} \angle 90^\circ \]
• **APPENDIX FILTER NETWORKS**

• WE HAVE FOUND THAT A PARALLEL RESONANT CIRCUIT HAS 
  1) A RESONANT FREQUENCY AT 
  \[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]
  2) A RESONANT IMPEDANCE OF \( R \)
  3) A \( Q = \frac{2\pi f_0 C R}{R} \)

• IF WE CONNECT 2 PARALLEL RESONANT CIRCUITS IN PARALLEL WE FIND

\[ L_{II} = \frac{L}{2} \]
\[ C_{II} = 2C \]
\[ R_{II} = \frac{R}{2} \]

\[ f_R = \frac{1}{2\pi \sqrt{\frac{L}{2} \times 2C}} = f_0 \text{ UNCHANGED} \]
\[ R' = \frac{R}{2} \]
\[ Q' = 2\pi f_0 \times 2C \times \frac{R}{2} = Q \text{ UNCHANGED} \]

IT YIELDED A LOWER IMPEDANCE AND NO CHANGE IN CURVE SHAPE.

• IF WE ISOLATE THE TWO FILTERS WITH A HIGH INPUT IMPEDANCE, LOW OUTPUT IMPEDANCE AMPLIFIER, WE CAN MULTIPLY THE RESPONSES OF THE CIRCUITS.
- **CASCADING FILTERS**

\[
\begin{align*}
H_1(\omega) & \quad 1:1 \\
H_2(\omega) & \quad H(\omega) = H_1(\omega) H_2(\omega) \quad \text{IF } H_1(\omega) = H_2(\omega) \\
\frac{H(\omega)}{\omega} & = \left[H_1(\omega)\right]^2
\end{align*}
\]

- **WE CAN NARROW THE BANDWIDTH OF THE FILTER BY CASCADING SEVERAL FILTER NETWORKS**

- **WE CAN ALSO BROADEN THE BANDWIDTH BY SHIFTING THE CENTER FREQUENCY OF ONE OR MORE OF THE FILTER NETWORKS. TO OBTAIN A FILTER WITH SHARP "SKIRTS" AND A BROAD BANDPASS**

\[
\begin{align*}
\theta = 100^\circ \\
\beta \omega & \approx 0.08 \frac{f}{f_0}
\end{align*}
\]

**STAGGER TUNED 4 STAGE PARALLEL RESONANT FILTER**
• Why do we need these filters?
  - In telephone touch tone dialing
  - In computer modem signal transmission
  - In radio receiver channel filtering

• Tone generation
  - A gated single frequency tone burst has more than one frequency component.

• Some spectral properties of time functions.
  - The infinite sine or cosine wave

\[ f(t) = A \cos \omega_0 t \]

- The presence of negative and positive frequencies comes from

\[ \cos \omega_0 t = \frac{e^{j \omega_0 t} + e^{-j \omega_0 t}}{2} \]

- The rect function

\[ g(t) = \text{RECT} \left( \frac{t}{T/2} \right) \]

\[ G(\omega) = \frac{AT}{2} \text{sinc} \left( \frac{\omega T}{2} \right) \]
- The spectrum of the cosine function multiplied by the rect function is a pair of sinc functions shifted to frequencies $+w_0$ and $-w_0$.

- If the pulse contains a large number of cycles, the sinc function is narrow in width compared to the center frequency $w_0$. 
  \[ \tau = \frac{1}{f_0} = \frac{w_0}{2\pi} \]  
  time for 1 cycle

- Example: $w_0 = 2\pi \times 10^3 \quad T = 10^{-3} \text{ sec} \quad \frac{2\pi}{2\pi \times 10^3} \tau < 10^{-1}$

- The width of the sinc function is $\propto \frac{1}{\tau}$

- The center frequency $w_0$ is 6280 rad/sec

- Thus the frequency spectrum looks like

- The filter required to pass this bandwidth would have a $Q = \frac{6280}{3} \approx 200 \quad w_0 = 6280$

- If the filter is narrower ($Q > 200$), part of the spectrum gets cut off and the pulse shape is distorted.
- MODEMS USE PULSE WIDTH MODULATION AND MULTIPLE FREQUENCY OPERATION IN ORDER TO SPEED UP THE TRANSMISSION OF INFORMATION. EACH PULSE WIDTH REQUIRES A SPECIAL "ACTIVE" FILTER TO PREVENT PULSE SHAPE DISTORTION.

- IN RADIO COMMUNICATION SYSTEMS THE SAME PRINCIPLES ARE APPLIED BUT AT MUCH HIGHER FREQUENCIES, i.e. MHZ VS KHZ.

- FOR HIGH RESOLUTION TV SYSTEMS VERY WIDE BANDWIDTHS ARE REQUIRED. THE AVAILABLE SPECTRUM FOR "ANTENNA" TV IS NOT LARGE ENOUGH TO SUPPORT HIGH DEFINITION TV. THEREFORE "SATELLITE" AND CABLE SYSTEMS HAVE BEEN USED TO GET THE NECESSARY BANDWIDTH.
A NUMERICAL EXAMPLE OF A TRANSFER FUNCTION WITH ISOLATED STAGES (CASCADE)

\[ V_{\text{out}_1} = V_{\text{in}_1} \frac{2 \omega}{2 + j \omega/\omega_{1/2}} = V_{\text{in}_1} \frac{1}{1 + \frac{j \omega}{\omega_{1/2}}} = V_{\text{in}_1} \frac{1}{1 - \frac{j \omega}{\omega_{1/2}}} \]

\[ V_{\text{out}_2} = V_{\text{out}_1} \times \frac{2 \omega}{2 + j \omega/50} = V_{\text{in}_1} \frac{1}{1 + \frac{j \omega}{100}} \cdot \frac{1}{1 - \frac{j \omega}{\omega_{1/2}}} \]

\[ H(\omega) = \frac{1}{1 + j \left( \frac{\omega}{100} - \frac{1}{\omega} \right) + \frac{1}{100}} \]

\[ |H(\omega)| = \left\| \frac{1}{\sqrt{\left(1 + \frac{1}{\omega_{100}}\right)^2 + \left(\frac{\omega}{\omega_{100}} - \frac{1}{\omega}\right)^2}} \right\| \]

<table>
<thead>
<tr>
<th>\omega</th>
<th>H(\omega)</th>
<th>H_{dB}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>-20</td>
</tr>
<tr>
<td>1.0</td>
<td>0.707</td>
<td>-3</td>
</tr>
<tr>
<td>10.0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
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</tr>
<tr>
<td>1000.0</td>
<td>0.1</td>
<td>-20</td>
</tr>
</tbody>
</table>

CUT OFF: \( \omega_A = 1 \), \( \omega_B = 100 \), \( \frac{\omega_B}{\omega_A} = 100 \) 2 DECADES
\[ H(\omega) = \frac{1}{(1 + \frac{1}{1000}) + j \left( \frac{\omega}{1000} - \frac{1}{\omega} \right)} \]

Linear amplitude
Decade frequency
(also log)

\[ \text{dB}(H(\omega)) \]

Log amplitude dB
Decade frequency
(also log)

Bode plot
What happens when the stages are not isolated?

\[
\begin{align*}
-V_{in} + \frac{1}{j\omega L} I_1 + 2(I_1 - I_2) &= 0 \\
(I_1 - I_2) L + j\omega \frac{L}{50} I_2 + 2I_2 &= 0 \\
I_1 (2 + \frac{2}{j\omega}) - I_2 (2) &= V_{in} \\
I_1 &= I_2 (2 + \frac{j\omega}{100})
\end{align*}
\]

Solving for \(V_{out}\)

\[
V_{out} = 2I_2 = \frac{V_{in}}{1 + \frac{1}{100} + j\left(\frac{\omega}{100} - \frac{2}{\omega}\right)}
\]

\[
H(\omega) = \frac{1}{\left(1 + \frac{1}{100}\right) + j\left(\frac{\omega}{100} - \frac{2}{\omega}\right)}
\]

Cut off: \(\omega_A = 2\), \(\omega_B = 100\), \(\frac{\omega_B}{\omega_A} = 50 = \angle \text{READ}\)

\[
|H(\omega)| = \frac{1}{\sqrt{(1 - \frac{\omega}{\omega_B})^2 + (\frac{\omega}{\omega_B} - \frac{2}{\omega})^2}}
\]

\[
\omega_A = \frac{1}{RC} = \frac{1}{2\sqrt{2}} = 1 \quad \omega_B = \frac{R}{C} = \frac{2}{150} = 100
\]
\[
\begin{align*}
\left[ -V_{in} + j\omega \frac{1}{50} I_1 + 2 (I_1 - I_2) = 0 \right] \\
\left[ (I_2 - I_1) \frac{1}{2} + \frac{1}{\omega} I_2 + 2I_2 = 0 \right] \\
\left[ I_1 \left( 2 + \frac{1}{\omega} \right) + I_2 (-2) = V_{in} \right] \\
\left[ I_1 (-2) + I_2 \left( 2 + \frac{1}{\omega} \right) = 0 \right]
\end{align*}
\]

\[
I_2 = \frac{V_{in}}{2 + j \left( \frac{2\omega}{50} + \frac{1}{\omega} + \frac{1}{50} \right)}
\]

\[
V_{out} = 2I_2 = \frac{V_{in}}{1 + \frac{1}{100} + j \left( \frac{\omega}{50} - \frac{1}{\omega} \right)}
\]

\[
|H(\omega)| = \frac{1}{\sqrt{\left(1 + \frac{1}{100}\right)^2 + \left(\frac{\omega}{50} - \frac{1}{\omega}\right)^2}}
\]

\[
W_A = \frac{R}{L} = \frac{2}{\frac{1}{50}} = 100 \quad W_B = \frac{1}{RC} = \frac{1}{2\sqrt{2}} = 1
\]

\[
W_A \text{ is altered by coupling to the second stage}
\]
Cascading two or more filter networks:

1. A stage of isolation between pairs of filters is desired.

2. The isolator is an amplifier with high input impedance and low output impedance. Ideally $Z_{input} = \infty$, $Z_{output} = 0$.

3. Operational amplifiers are multiple-transistor packages which provide these properties via feedback circuitry. We will cover them soon.

4. The characteristic transfer function of each isolated filter can be multiplied to get the cascaded transfer function.

5. If this is done in dB's the responses of the filters can be added at each decade frequency.

<table>
<thead>
<tr>
<th>$f$ (Hz)</th>
<th>0.1</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>1,000</th>
<th>10,000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Filt. #1</td>
<td>-6</td>
<td>-3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>Filt. #2</td>
<td>-10</td>
<td>-5</td>
<td>0</td>
<td>0</td>
<td>-5</td>
<td>-10</td>
</tr>
</tbody>
</table>

$H(w)_{total} = -16 - 80 0 0 -5 -16$
Cross Over Network

\[ Z_1 = R_1 + j \omega L \]

\[ Z_2 = R_2 + \frac{1}{j \omega C} \]

\[ R_5 = R_1 = R_2 \]

\[ \frac{1}{C} = R_8 \quad \omega_0^2 = \frac{1}{LC} \]

\[ \frac{Z_1}{Z_2} = Z_L = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{(R + j \omega L)(R - j \omega C)}{2R + \frac{1}{j \omega C}} \]

\[ = \frac{(R^2 + \frac{1}{\omega C}) + j R (\omega L - \frac{1}{\omega C})}{2R + \frac{1}{j \omega C}} \]

For \( \frac{1}{C} = R_5 \Rightarrow R_1 = R_2 = R_8 \neq R_2 \)

\[ Z_L = R_5 \quad \text{All Pass Network} \]

For \( L = \frac{4}{1000} \quad C = \frac{1}{64000} \quad \omega_0 = 4000 = \frac{1}{\sqrt{LC}} \)

\[ \frac{1}{C} = (16)^2 \]

\[ @ \omega = \omega_0 \quad \frac{1}{\omega C} = 16 \quad \omega L = 16 \]

If \( V_S = 20 \, V \)

\[ Z_{AB} = Z_{11} = 16 \, \pi \quad \Rightarrow R_5 = 16 \, \pi \]

\[ R_1 = R_2 = 16 \, \pi \]

\[ V_{AB} = 20 \times \frac{16}{16 + 16} = 10 \quad \text{for all } \omega \]

\[ V_{R_1} = \frac{V_{AB} \times R_1}{Z_1} = \frac{10 \times 16}{16 + 316} = \frac{160}{16\sqrt{2}} \]

\[ V_{R_2} = \frac{V_{AB} \times R_2}{Z_2} = \frac{10 \times 16}{16 + 16} = \frac{160}{16 \sqrt{2}} \]
\[ w = \omega_0 \]
\[ P_{1\text{ RMS}} = \frac{(V_{R1})^2}{2R_1} = \frac{100}{2 \times 2 \times 16} = \frac{25}{16} \quad P_{2\text{ RMS}} = \frac{(V_{R2})^2}{2R_2} = \frac{25}{16} \]

\[ w = 0 \quad j\omega L = 0 \quad \frac{1}{j\omega C} = -j\infty \]
No current in \( R_1 \)

\[ V_{R1}(0) = 20 \times \frac{16}{16 + 16} = 10 \]
\[ P_{1\text{ RMS}}(0) = \left( \frac{V_{R1}(0)}{2R_1} \right)^2 = \frac{100}{2 \times 16} = \frac{25}{16} \]

\[ w = \infty \quad j\omega L = j\infty \quad \frac{1}{j\omega C} = 0 \]
No current in \( R_1 \)

\[ V_{R2}(\infty) = 20 \times \frac{16}{16 + 16} = 10 \]
\[ P_{2\text{ RMS}}(\infty) = \left( \frac{V_{R2}(\infty)}{2R_2} \right)^2 = \frac{100}{2 \times 16} = \frac{25}{16} \]

At crossover, \( P_1 + P_2 = \frac{50}{16} \) \( \omega \)

\( \omega = 0 \)
\[ P_1 = \frac{50}{16} \quad P_2 = 0 \]

\( \omega = \infty \)
\[ P_2 = \frac{50}{16} \quad P_1 = 0 \]

\( \Phi \in D_1 \)
\[ V_{R1}(\omega) = \frac{V_{AB} \times R_1}{Z_1} = \frac{10 \times R_1}{R_1 + j \omega L} \]

Using \( \omega_0^2 = \frac{1}{LC} \)

\[ V_{R1}(\omega) = \frac{10}{1 + j \frac{\omega L}{R}} \]

But \( L = \frac{1}{\omega_0^2 C} \)

\[ = \frac{10}{1 + j \frac{\omega}{R} \frac{\omega_0}{\omega_0^2 C}} = \frac{10}{1 + j \left( \frac{1}{\omega_0^2 C} \right) \left( \frac{\omega}{\omega_0} \right)} \]

Since \( \omega_0 \approx 20 \)

\[ Q_s = \frac{1}{\omega_0 CR} \]

\[ \frac{V_{R1}(\omega)}{V_s} = \frac{1}{2} \left( 1 + j Q_s \left( \frac{\omega}{\omega_0} \right) \right) \]

\[ |H_{R1}(\omega)| = \frac{10}{2 \sqrt{1 + (Q_s \left( \frac{\omega}{\omega_0} \right))^2}} \]

\[ V_{R2}(\omega) = \frac{10}{1 + j \omega CR} \]

\[ H_{R2}(\omega) = \frac{1}{2 \sqrt{1 - j \left( \frac{Q_s \left( \frac{\omega}{\omega_0} \right)}{1 + Q_s \left( \frac{\omega}{\omega_0} \right)^2} \right)}} \]

\[ H_{R2}(\omega) = \frac{1}{2 \sqrt{1 + (Q_s \left( \frac{\omega}{\omega_0} \right))^2}} \]

\[ Q_s = \frac{1}{\omega_0 CR} \]

\[ = \frac{1}{4000 \times \frac{1}{4000} \times 16} \]

\[ Q_s = 1 \]

Plot for

\[ \omega = 0.01 \omega_0 \]

\[ 0.1 \omega_0 \]

\[ 1 \omega_0 \]

\[ 10 \omega_0 \]

\[ 100 \omega_0 \]
Using dB Scale

\[ |H_{r1}(w)|_{dB} = 20 \log_{10} 1 - 20 \log_{10} 2 - 20 \times \frac{1}{2} \log_{10} |H_{r1}(w)| \]

\[ |H_{r2}(w)|_{dB} = 20 \log_{10} 1 - 20 \log_{10} 2 - 20 \times \frac{1}{2} \log_{10} |H_{r2}(w)| + (\text{constant}) \]

| \( w/w_0 \) | \( |H_{r1}(w)|_{dB} \) | \( |H_{r2}(w)|_{dB} \) |
|-------------|------------------|------------------|
| 0.01        | -6               | -4.6             |
| 0.10        | -6               | -4.6             |
| 1.00        | -6               | -6               |
| 10.00       | -4.6             | -6               |
| 100.00      | -4.6             | -6               |

Bode Plot of Crossover Response