1. (10 PTS) Is the system BIBO stable?
From the figure, the system difference equation is given by
\[ y(n) - \frac{3}{4} y(n - 1) + \frac{1}{8} y(n - 2) = x(n), \]
so that its characteristic equation is:
\[ \lambda^2 - \frac{3}{4} \lambda - \frac{1}{8} = 0 \]
\[ \left( \lambda - \frac{1}{2} \right) \left( \lambda - \frac{1}{4} \right) = 0 \]
Both modes of the system (\( \lambda = \frac{1}{2} \) and \( \lambda = \frac{1}{4} \)) are inside the unit disc. Since the system is causal, then it is BIBO stable.

2. (30 PTS) Find the complete response of the system when \( x(n) = \alpha^n u(n) \), where \(|\alpha| < 1\).

**Particular solution**
For \( x(n) = \alpha^n u(n) \), the particular solution has the form \( y_p(n) = K\alpha^n u(n) \). To evaluate \( K \), we substitute \( y_p(n) \) into the difference equation
\[ K\alpha^n u(n) - \frac{3}{4} K\alpha^{n-1} u(n - 1) + \frac{1}{8} K\alpha^{n-2} u(n - 2) = \alpha^n u(n) \]
For \( n \geq 2 \),
\[ K\alpha^2 - \frac{3}{4} K\alpha + \frac{1}{8} K = \alpha^2 \]
so that
\[ K = \frac{\alpha^2}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} \]
and, hence,
\[ y_p(n) = \frac{\alpha^n}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} u(n), \quad n \geq 2 \]

**Homogeneous solution**
\[ y_h(n) = C_1 \left( \frac{1}{2} \right)^n + C_2 \left( \frac{1}{4} \right)^n \]
Then, the **complete solution** $y(n)$ is given by:

$$y(n) = y_h(n) + y_p(n) = C_1 \left( \frac{1}{2} \right)^n + C_2 \left( \frac{1}{4} \right)^n + \frac{\alpha^{n+2}}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} u(n), \quad n \geq 2$$

Using the system difference equation and the initial conditions $y(-2) = 0$ and $y(-1) = -\frac{4}{3}$, we get

$$y(0) = \frac{3}{4} y(-1) - \frac{1}{8} y(-2) + x(0) = 0$$

$$y(1) = \frac{3}{4} y(0) - \frac{1}{8} y(-1) + x(1) = \alpha + \frac{1}{6}$$

Then $C_1$ and $C_2$ should satisfy

$$C_1 + C_2 + \frac{\alpha^2}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} = y(0) = 0$$

$$\frac{1}{2} C_1 + \frac{1}{4} C_2 + \frac{\alpha^3}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} = y(1) = \alpha + \frac{1}{6}$$

solving for $C_1$ and $C_2$, we get

$$C_1 = -\frac{\frac{4}{3} \alpha^2 - \frac{1}{12}}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} = -\frac{4}{3} \alpha + \frac{1}{2}$$

$$C_2 = \frac{\frac{1}{3} \alpha^2 - \frac{1}{12}}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} = \frac{1}{3} \alpha - \frac{1}{4}$$

It follows that

$$y(n) = \frac{1}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} \left[ -\left( \frac{4}{3} \alpha^2 - \frac{1}{12} \right) \left( \frac{1}{2} \right)^n + \left( \frac{1}{3} \alpha^2 - \frac{1}{12} \right) \left( \frac{1}{4} \right)^n + \alpha^{n+2} \right] u(n), \quad n \geq 0$$

3. **(10 PTS)** Are there choices of $\alpha$ for which at least one of the modes of the system is not excited (i.e., does not appear) at the output? Describe all such $\alpha$'s.

For the mode $\lambda = \frac{1}{2}$ to disappear, we choose $\alpha$ such that $u$, i.e.

$$\alpha = -\frac{1}{4}$$

Similarly, for the mode $\lambda = \frac{1}{4}$ to disappear, we choose $\alpha$ such that $C_2 = 0$, i.e.

$$\alpha = -\frac{1}{2}$$

4. **(10 PTS)** Find the energy of the output sequence when $\alpha = -1/4$.

$$y(n) = \frac{8}{3} \left[ \left( \frac{1}{16} \right)^n + \left( \frac{1}{4} \right)^{n+2} \right] u(n)$$

$$= \frac{1}{6} \left[ \left( \frac{-1}{4} \right)^n - \left( \frac{1}{4} \right)^n \right] u(n)$$

$$= \frac{1}{3} \left( \frac{-1}{4} \right)^{2n+1} u(n)$$

$$= -\frac{1}{12} \left( \frac{-1}{4} \right)^{2n} u(n)$$
The energy of the sequence \( y(n) \) is given by

\[
E_Y = \sum_{n=\infty}^{\infty} |y(n)|^2
\]

\[
= \left( \frac{1}{12} \right)^2 \sum_{n=0}^{\infty} \left( \frac{-1}{4} \right)^{4n}
\]

\[
= \frac{1}{144} \sum_{n=0}^{\infty} \left( \frac{1}{256} \right)^n
\]

\[
= \frac{1}{144} \cdot \frac{1}{1 - \frac{1}{256}} = 0.007
\]

5. (30 PTS) Find the same complete response as in part 2) above by using the \( z \)-transform technique.

**Zero–state solution**

Taking the \( Z \)-transform of the difference equation, we find that the \( Z \)-transform of the zero–state solution is

\[
Y_{zs}(z) - \frac{3}{4} z^{-1} Y_{zs}(z) + \frac{1}{8} z^{-2} Y_{zs}(z) = X(z)
\]

\[
Y_{zs}(z) = \frac{X(z)}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}} = \frac{z^2 X(z)}{(z - \frac{1}{4})(z - \frac{1}{2})}
\]

For \( x(n) = \alpha^n u(n) \)

\[
X(z) = \frac{z}{z - \alpha}, \quad |z| > |\alpha|
\]

Then,

\[
Y_{zs}(z) = \frac{z^3}{(z - \frac{1}{4})(z - \frac{1}{2})(z - \alpha)}
\]

Define \( \tilde{Y}(z) = z^{-1} Y_{zs}(z) \). Using partial fractions, \( \tilde{Y}(z) \) can be written as

\[
\tilde{Y}(z) = \frac{z^2}{(z - \frac{1}{4})(z - \frac{1}{2})(z - \alpha)}
\]

\[
= \frac{A}{z - \frac{1}{4}} + \frac{B}{z - \frac{1}{2}} + \frac{C}{z - \alpha}
\]

The constants \( A, B, \) and \( C \) are evaluated as follows

\[
A = \left( z - \frac{1}{4} \right) \tilde{Y}(z)|_{z=\frac{1}{4}} = \frac{1}{\alpha - \frac{1}{4}}
\]

\[
B = \left( z - \frac{1}{2} \right) \tilde{Y}(z)|_{z=\frac{1}{2}} = \frac{-1}{\alpha - \frac{1}{2}}
\]

\[
C = (z - \alpha) \tilde{Y}(z)|_{z=\alpha} = \frac{\alpha^2}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}}
\]

Substituting \( A, B, \) and \( C \), we get

\[
\tilde{Y}(z) = \frac{\frac{1}{4} \cdot \frac{1}{z - \frac{1}{4}} - \frac{1}{4} \cdot \frac{1}{z - \frac{1}{2}} + \frac{\alpha^2}{\alpha^2 - \frac{3}{4} \alpha + \frac{1}{8}} \cdot \frac{1}{z - \alpha}}
\]
\[ y_{zs}(z) = \frac{1}{2} \cdot \frac{z}{z-\frac{1}{2}} + \frac{-1}{2} \cdot \frac{z}{z-\frac{3}{4}} + \frac{\alpha^2}{4} \cdot \frac{1}{z-\frac{1}{4}} \]\[ = \frac{1}{a^2 - \frac{3}{4} \alpha + \frac{1}{8}} \left[-\left(\frac{1}{4}\right) \cdot \frac{z}{z-\frac{1}{2}} + \frac{1}{4} \left(\frac{1}{2}\right) \cdot \frac{z}{z-\frac{1}{4}} + a^2 \cdot \frac{z}{z-\alpha}\right] \]

Using inverse Z-transform, \( y_{zs}(n) \) is

\[ y_{zs}(n) = \frac{-2}{a^2 - \frac{3}{4} \alpha + \frac{1}{8}} \left[-\left(\frac{1}{4}\right) \left(\frac{1}{2}\right)^n + \frac{1}{4} \left(\frac{1}{2}\right) \left(\frac{1}{4}\right)^n + a^n \right] u(n) \]

**Zero–input solution**

\[ y_{zi}(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{4}\right)^n \]

Using the initial conditions \( y(-1) = -\frac{4}{3} \) and \( y(-2) = 0 \) to determine \( C_1 \) and \( C_2 \) we get

\[ 2C_1 + 4C_2 = -\frac{4}{3} \]
\[ 4C_1 + 16C_2 = 0 \]

By solving these two equations, we find \( C_1 = -\frac{4}{3} \) and \( C_2 = \frac{1}{3} \). Then,

\[ y_{zi}(n) = -\frac{4}{3} \left(\frac{1}{2}\right)^n + \frac{1}{3} \left(\frac{1}{4}\right)^n \]

**Complete solution**

Adding the zero–input solution to the zero–state solution we obtain the complete solution

\[ y(n) = y_{zs}(n) + y_{zi}(n) \]

\[ = \frac{1}{a^2 - \frac{3}{4} \alpha + \frac{1}{8}} \left[-\left(\frac{1}{4}\right) \left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right) \left(\frac{1}{2}\right) \left(\frac{1}{4}\right)^n + a^n \right] u(n), \quad n \geq 0 \]

6. (10 BONUS PTS) Which value of \( \alpha \) results in an output sequence with smallest energy?

Let \( y(n) = [C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{4}\right)^n + Ka^n] u(n) \) where

\[ C_1 = -\frac{4}{3} \alpha^2 - \frac{1}{12} \]
\[ C_2 = \frac{4}{3} \alpha^2 - \frac{1}{12} \]
\[ K = \frac{\alpha^2}{a^2 - \frac{3}{4} \alpha + \frac{1}{8}} \]

Then,

\[ E_Y = \sum_{n=-\infty}^{\infty} |y(n)|^2 \]

\[ = \sum_{n=-\infty}^{\infty} C_1^2 \left(\frac{1}{2}\right)^{2n} + C_2^2 \left(\frac{1}{4}\right)^{2n} + K^2 \alpha^{2n} + 2C_1 C_2 \left(\frac{1}{8}\right)^n + 2C_1 K \left(\frac{\alpha}{2}\right)^n + 2C_2 K \left(\frac{\alpha}{4}\right)^n \]

\[ = \frac{4}{3} C_1^2 + \frac{16}{15} C_2^2 + \frac{K^2}{1-\alpha^2} + \frac{16}{1-\alpha^2} C_1 C_2 + \frac{2C_1 K}{1-\frac{\alpha}{2}} + \frac{2C_2 K}{1-\frac{\alpha}{4}} \]

Then we find the value of \( \alpha \) results in an output sequence with smallest energy by setting \( \frac{d}{d\alpha} E_Y = 0 \) and solving the resulting equation.