5.3. Let
\[ J = \frac{400 \sin \theta}{r^2 + 4} \, a_r \, \text{A/m}^2 \]

a) Find the total current flowing through that portion of the spherical surface \( r = 0.8 \), bounded by \( 0.1 \pi < \theta < 0.3 \pi, 0 < \phi < 2 \pi \): This will be
\[ I = \int \int J \cdot n \, da = \int_0^{2\pi} \int_0^{\pi} \frac{400 \sin \theta}{(8)^2 + 4(8)^2} \sin \theta \, d\theta \, d\phi = 400 \frac{(8)^2 \pi}{464} \int_0^{\pi} \sin^2 \theta \, d\theta \]
\[ = 346.5 \int_0^{\pi} \frac{1}{2} [1 - \cos(2\theta)] \, d\theta = 77.4 \, \text{A} \]

b) Find the average value of \( J \) over the defined area. The area is
\[ \text{Area} = \int_0^{2\pi} \int_0^{\pi} (8)^2 \sin \theta \, d\theta \, d\phi = 1.46 \, \text{m}^2 \]

The average current density is thus \( J_{\text{avg}} = \frac{(77.4/1.46)}{a_r} = 53.0 \, \text{A/m}^2 \).

5.5. Let
\[ J = \frac{25}{\rho} \, a_p - \frac{20}{\rho^2 + 0.01} \, a_z \, \text{A/m}^2 \]

a) Find the total current crossing the plane \( z = 0.2 \) in the \( a_z \) direction for \( \rho < 0.4 \): Use
\[ I = \int \int J \cdot n \, |z=0.2| \, da = \int_0^{2\pi} \int_0^{\pi} \frac{-20}{\rho^2 + 0.01} \, \rho \, d\rho \, d\phi \]
\[ = -\frac{1}{2} \int_0^{2\pi} [20 \ln(\rho^2 + 0.01)] |z=0.2| \, d\phi = -20 \pi \ln(17) = -178.0 \, \text{A} \]

b) Calculate \( \partial \rho \rho / \partial t \): This is found using the equation of continuity:
\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot J = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho \rho_J) + \frac{\partial J_z}{\partial z} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (25) + \frac{\partial}{\partial z} \left( \frac{-20}{\rho^2 + 0.01} \right) = 0 \]

c) Find the outward current crossing the closed surface defined by \( \rho = 0.01, \rho = 0.4, z = 0, \) and \( z = 0.2 \): This will be
\[ I = \int_0^{2\pi} \int_0^{0.01} a_p \cdot (-a_p) \, d\phi \, dz + \int_0^{2\pi} \int_0^{0.4} a_p \cdot (a_p) \, d\phi \, dz \]
\[ + \int_0^{2\pi} \int_0^{0.01} a_z \cdot (-a_z) \, \rho \, d\rho \, d\phi + \int_0^{2\pi} \int_0^{0.4} a_z \cdot (a_z) \, \rho \, d\rho \, d\phi = 0 \]
since the integrals will cancel each other.

d) Show that the divergence theorem is satisfied for \( J \) and the surface specified in part b. In part c, the net outward flux was found to be zero, and in part b, the net flux of \( J \) was found to be zero (as will be its volume integral). Therefore, the divergence theorem is satisfied.
5.12. Two identical conducting plates, each having area $A$, are located at $z = 0$ and $z = d$. The region between plates is filled with a material having $z$-dependent conductivity, $\sigma(z) = \sigma_0 e^{-z/\lambda}$, where $\sigma_0$ is a constant. Voltage $V_0$ is applied to the plate at $z = d$; the plate at $z = 0$ is at zero potential. Find, in terms of the given parameters:

a) the resistance of the material: We start with the differential resistance of a thin slab of the material of thickness $dz$, which is

$$dR = \frac{dz}{\sigma A} = \frac{e^{-z/\lambda}dz}{\sigma_0 A} \text{ so that } R = \int_0^d dR = \int_0^d \frac{e^{-z/\lambda}dz}{\sigma_0 A} = \frac{d}{\sigma_0 A} (e^{-1}) = \frac{1.72d}{\sigma_0 A} \Omega$$

b) the total current flowing between plates: We use

$$I = \frac{V_0}{R} = \frac{\sigma_0 A V_0}{1.72d}$$

c) the electric field intensity $E$ within the material: First the current density is

$$J = \frac{I}{A} a_z = -\frac{\sigma_0 V_0}{1.72d} a_z \text{ so that } E = \frac{J}{\sigma(z)} = -\frac{V_0 e^{-z/d}}{1.72d} a_z \text{ V/m}$$

5.15. Let $V = 10(\rho + 1) z^2 \cos \phi$ V in free space.

a) Let the equipotential surface $V = 20$ V define a conductor surface. Find the equation of the conductor surface: Set the given potential function equal to 20, to find:

$$\rho + 1 = \frac{20}{z^2 \cos \phi}$$

b) Find $\rho$ and $E$ at that point on the conductor surface where $\phi = 0.2\pi$ and $z = 1.5$: At the given values of $\phi$ and $z$, we solve the equation of the surface found in part a for $\rho$, obtaining $\rho = 10$. Then

$$E = -\nabla V = -\frac{\partial V}{\partial \rho} \hat{a}_\rho - \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{a}_\phi - \frac{\partial V}{\partial z} \hat{a}_z = -10z^2 \cos \phi \hat{a}_\rho + 10 \frac{\rho + 1}{\rho} z^2 \sin \phi \hat{a}_\phi - 20(\rho + 1) z \cos \phi \hat{a}_z$$

Then

$$E(10, 2\pi, 1.5) = -18.2 \hat{a}_\rho + 145 \hat{a}_\phi - 26.7 \hat{a}_z \text{ V/m}$$

c) Find $|\rho_s|$ at that point: Since $E$ is at the perfectly-conducting surface, it will be normal to the surface, so we may write:

$$\rho_s = \varepsilon_0 E \left| \frac{\mathbf{E} \cdot \mathbf{n}}{|\mathbf{E}|} \right|_{\text{surface}} = \varepsilon_0 \frac{E \cdot E}{|E|} = \varepsilon_0 \sqrt{E \cdot E} = \varepsilon_0 \sqrt{(18.2)^2 + (145)^2 + (26.7)^2} = 1.32 \text{ nC/m}^2$$
5.22. The line segment $x = 0$, $-1 \leq y \leq 1$, $z = 1$, carries a linear charge density $\rho_L = \pi |y| \mu C/m$. Let $z = 0$ be a conducting plane and determine the surface charge density at: (a) $(0,0,0)$; (b) $(0,1,0)$.

We consider the line charge to be made up of a string of differential segments of length, $dy'$, and of charge $dq = \rho_L\, dy'$. A given segment at location $(0, y', 1)$ will have a corresponding image charge segment at location $(0, y', -1)$. The differential flux density on the $y$ axis that is associated with the segment-image pair will be

$$dD = \frac{\rho_L}{4\pi} \frac{(y - y') a_y - a_z}{((y - y')^2 + 1)\sqrt{2}} - \frac{\rho_L}{4\pi} \frac{(y - y') a_y + a_z}{((y - y')^2 + 1)\sqrt{2}}$$

In other words, each charge segment and its image produce a net field in which the $y$ components have cancelled. The total flux density from the line charge and its image is now

$$D(y) = \int dD = \int_{-1}^{1} \frac{-\pi |y'| a_y \, dy'}{2\pi ((y - y')^2 + 1)^{3/2}}$$

$$= \left. \frac{a_z}{2} \int_{0}^{1} \left[ \frac{y'}{((y - y')^2 + 1)^{3/2}} + \frac{y'(y + y') + 1}{((y + y')^2 + 1)^{3/2}} \right] \right|_{0}^{1}$$

$$= \left. \frac{a_z}{2} \left[ \frac{y(y - 1) + 1}{((y - 1)^2 + 1)^{1/2}} + \frac{y(y + 1) + 1}{((y + 1)^2 + 1)^{1/2}} - 2(y^2 + 1)^{1/2} \right] \right|_{0}^{1}$$

Now, at the origin (part $a$), we find the charge density through

$$\rho_L(0,0,0) = D \cdot a_z \bigg|_{y=0} = \frac{a_z}{2} \left[ \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 2 \right] = -0.29 \mu C/m^2$$

Then, at $(0,1,0)$ (part $b$), the charge density is

$$\rho_L(0,1,0) = D \cdot a_z \bigg|_{y=1} = \frac{a_z}{2} \left[ 1 + \frac{3}{\sqrt{5}} - 2 \right] = -0.24 \mu C/m^2$$

5.24. At a certain temperature, the electron and hole mobilities in intrinsic germanium are given as $0.43$ and $0.21$ m$^2$/Vs, respectively. If the electron and hole concentrations are both $2.5 \times 10^{19}$ m$^{-3}$, find the conductivity at this temperature.

With the electron and hole charge magnitude of $1.6 \times 10^{-19}$ C, the conductivity in this case can be written:

$$\sigma = |\rho_L| \mu_e + |\rho_L| \mu_h = (1.6 \times 10^{-19})(2.3 \times 10^{19})(0.43 + 0.21) = 2.36 \text{ S/m}$$