Prob. 1 Consider the linear transformation $A$ on the vector space $V$ with basis $\mathcal{B} = \{v_1, v_2, v_3\}$ defined by

$$
A v_1 = v_1 - v_3,
A v_2 = v_1 + v_2,
A v_3 = v_1 + v_3.
$$

(a) Let $V = \mathbb{R}^3$ (the 3-dimensional real coordinate space). Determine all the eigenvalues of $A$ and their corresponding eigenvectors. Is $A$ a simple linear transformation?

(b) Repeat (a) with $V$ replaced by $\mathbb{C}^3$ (the 3-dimensional complex coordinate space).

(c) For $V = \mathbb{R}^3$, determine the rank and nullity of $A$. Are your answers altered if we replace $V$ by $\mathbb{C}^3$?

Prob. 2 The equations of motion of a rigid spacecraft with applied control torque (e.g. gas jet control) in terms of the components of its angular rates in the principal axes of the spacecraft are given by

$$
I_1 \frac{d\omega_1}{dt} = - (I_3 - I_2) \omega_2 \omega_3 + u_1(t),
$$

$$
I_2 \frac{d\omega_2}{dt} = - (I_1 - I_3) \omega_3 \omega_1 + u_2(t),
$$

$$
I_3 \frac{d\omega_3}{dt} = - (I_2 - I_1) \omega_1 \omega_2 + u_3(t),
$$

where $I_i$, $\omega_i$, $u_i$, $i = 1, 2, 3$, are the moment of inertia, the component of the angular velocity, and the applied torque respectively, all with respect to the $i$-th principal axis of the spacecraft.

(a) Rewrite the above equations in normal form (i.e. $dx/dt = f(x,u(t))$).

(b) Let $I_1 = I_2 = I$. Show that the solution to the above equations with $u_1(t) = u_2(t) = u_3(t) = 0$ for all $t$ and initial conditions $\omega_i(0) = \omega_{i0}$, $i = 1, 2, 3$, has the form:

$$
\omega_1(t) = \omega_0 \sin(\beta t + \alpha), \quad \omega_2(t) = \omega_0 \cos(\beta t + \alpha), \quad \omega_3(t) = \omega_{30},
$$

where

$$
\omega_0 = \left[ \omega_{10}^2 + \omega_{20}^2 \right]^{1/2}; \quad \alpha = \tan^{-1}\left( \frac{\omega_{20}}{\omega_{10}} \right), \quad \beta = \frac{I_1 - I_3}{I_1} \omega_{30}.
$$

(c) Let $\delta \omega_1(t), \delta \omega_2(t)$ and $\delta \omega_3(t)$ be the perturbed angular rates about the trajectory specified by (*). Derive the linearized equations for $\delta \omega_i(t)$, $i = 1, 2, 3$, in normal form with controls $u_i(t) = \delta u_i(t)$, $i = 1, 2, 3$.

NOTE: At this time, you should have a pretty good idea on the system you wish to consider for the mini-project. If you have already chosen the system, give one paragraph description of your system.