PROBLEM SET 4

Prob. 1 Let \( B = \{v_1, v_2, v_3\} \) be a basis for \( \mathbb{R}^3 \), and \( A \) be a linear transformation on \( \mathbb{R}^3 \) into \( \mathbb{R}^3 \) defined by

\[
\begin{align*}
Av_1 &= 4v_2, \\
Av_2 &= -v_1 + v_3, \\
Av_3 &= v_2.
\end{align*}
\]

(a) Determine whether \( A \) is a simple linear transformation.

(b) What is the dimension of the subspace spanned by the eigenvectors of \( A \)?

(c) Repeat (a) and (b) for \( A^2 \).

Prob. 2 Show that if \( A \) is a linear transformation on a vector space \( V \) into \( V \) such that

\[A^3 - 3A^2 + 3A - 2I = 0\]

where \( I \) is the identity transformation on \( V \), then \( A \) is nonsingular. Find \( A^{-1} \) in terms of \( A \).

Prob. 3 For a linear transformation \( B \) on \( \mathbb{R}^3 \) into \( \mathbb{R}^3 \) whose matrix representation with respect to some basis \( B \) is given by

\[
[B]_B = \frac{1}{8} \begin{bmatrix}
9 & 0 & -3 \\
10 & -8 & 2 \\
3 & 0 & -1
\end{bmatrix}.
\]

(a) Find

(i) the characteristic polynomial;

(ii) the eigenvalues and their corresponding eigenvectors;

(iii) the determinant and trace.

(b) Repeat (a) for the linear transformation \( B' \) whose matrix representation with respect to basis \( B \) is \([B]_B^T\) (the transpose of \([B]_B\)).

Prob. 4 Consider the linear transformation \( A \) on \( \mathbb{R}^3 \) into \( \mathbb{R}^3 \) whose matrix representation with respect to some basis is given by

\[
[A] = \begin{bmatrix}
1 & 1/2 & 1/2 \\
0 & 2 & 0 \\
-2 & 1 & 3
\end{bmatrix}.
\]

(a) Write down the characteristic polynomial of \( A \).

(b) Determine the spectrum and the eigenvectors of \( A \).

(c) Determine the algebraic and geometric multiplicities of each eigenvalue. Is \( A \) simple?

Prob. 5 Consider the linear transformation \( A \) defined on \( \mathbb{R}^3 \) defined by

\[
\begin{align*}
Av_1 &= v_1 - v_2, \\
Av_2 &= -v_1 + v_2 + v_3, \\
Av_3 &= -v_1 + v_2,
\end{align*}
\]

where \( B = \{v_1, v_2, v_3\} \) is a basis for \( \mathbb{R}^3 \).

(a) Determine the spectrum of \( A \).

(b) Determine all the eigenvectors of \( A \) (express them in terms of the
basis vectors in $\mathcal{B}$). Show that the eigenvectors of $A$ form a basis for the space $\mathbb{R}^3$. Write down the matrix representation of $A$ with respect to its eigenvectors.

(c) Find the linear transformation $B$ which relates the eigenvectors of $A$ and the basis vectors in $\mathcal{B}$.

(d) Find the matrix representation of $C = A^5 + 3A^3$ with respect to basis $\mathcal{B}$. 