Problem 1 (a) \( x(k+2) + 2x(k+1) - 3x(k) = 0 \).

Seek a solution of the form \( x(k) = \lambda^k \). The characteristic equation is:
\[ \lambda^2 + 2\lambda - 3 = 0. \]

Thus, the general solution is:
\[ x(k) = c_1(-3)^k + c_2. \]

To satisfy the initial conditions: \( x(0) = c_1 + c_2, \quad x(1) = -3c_1 + c_2 \Rightarrow c_1 = (x(0) - x(1))/4 \) and \( c_2 = (3x(0) + x(1))/4. \) If \( x(1) = x(0) \neq 0 \), then \( c_1 = 0 \) and \( x(k) = c_2 \) for all \( k \geq 0 \).

ANS. True.

(b) Consider:
\[ \alpha(v_1 - 2v_2 + v_3) + \beta(v_1 - 2v_2 + v_3) + \gamma(2v_1 + v_3) = 0. \]
\[ (\alpha + \beta + \gamma)v_1 - 2(\alpha + \beta)v_2 + (\alpha - \beta + \gamma)v_3 = 0. \]

By linear independence of \( \{v_1, v_2, v_3\} \), we have
\[ \begin{cases} 
\alpha + \beta + 2\gamma = 0, \\
-2(\alpha + \beta) = 0, \\
\alpha - \beta + \gamma = 0.
\end{cases} \]

Hence, \( \{w_1, w_2, w_3\} \) is also linearly independent.

ANS. True.

(c) Given:
\[ \begin{align*}
x_1(k+1) &= x_1(k) - x_2(k) + 3, \\
x_2(k+1) &= -x_1(k) + 2x_2(k) - 2.
\end{align*} \tag{1} \tag{2}

we have
\[ x_1(k+2) = x_1(k+1) - x_2(k+1) + 3. \tag{3} \]

Using (2), (3) can be rewritten as
\[ x_1(k+2) = x_1(k+1) + x_1(k) - 2x_2(k) + 5. \tag{4} \]

Using (1) to eliminate \( x_2(k) \) in (4), we obtain
\[ x_1(k+2) = x_1(k+1) + x_1(k) - 2(x_1(k) - x_1(k+1) + 3) + 5 \\
= 3x_1(k+1) - x_1(k) + 1. \]

ANS. False

(d) The set \( S \) of vectors \( x = (x_1, x_2, x_3) \) in \( R^3 \) such that \( x_1 + x_2 = 0 \) and \( x_3 \neq 0 \) is a subspace of \( R^3 \), since \( S = \{ x \in R^3 : x = (x_1, -x_1, x_3) = x_1(1, -1, 1), \}
\( x_3 \) is any real number). But \( S \) is one-dimensional, since \( v = (1, -1, 1) \) is a basis for \( S \).

ANS. False

(e) By assumption: \( Av = \lambda v, \) \( B = \mu v. \) Thus, \( (A^3 B^2 - 2B)v = A^3 B^2(\lambda v) - 2\mu v = \lambda A^3 B^2 v - 2\mu v = \lambda^2 A^3 v - 2\mu v = (\lambda^2 \lambda - 2\mu)v = (\lambda^4 \mu - 2\mu)v. \) Therefore, \( v \) is an eigenvector of \( C = A^3 B^2 - 2B \) corresponding to the eigenvalue \( \lambda^4 \mu - 2\mu. \)

When \( \lambda^4 \mu = 2\mu, \) then \( C \) is singular.

ANS: True.

(f) From the matrix representation of \( A \) with respect to \( B \), we have
\[ \mathbf{A} \mathbf{v}_1 = -2 \mathbf{v}_1 + 2 \mathbf{v}_2 + \mathbf{v}_3, \]
\[ \mathbf{A} \mathbf{v}_2 = -2 \mathbf{v}_2 + 2 \mathbf{v}_3, \]
\[ \mathbf{A} \mathbf{v}_3 = -2 \mathbf{v}_1 + \mathbf{v}_3. \]

Thus,
\[ \mathbf{A}(\mathbf{v}_1 + 2 \mathbf{v}_2) = -2 \mathbf{v}_1 + 2 \mathbf{v}_2 + \mathbf{v}_3 + 2(- \mathbf{v}_2 + 2 \mathbf{v}_3) = -\mathbf{v}_1 - 4 \mathbf{v}_2 + 5 \mathbf{v}_3, \]
\[ \mathbf{A} \mathbf{v}_2 = -2 \mathbf{v}_2 + 2 \mathbf{v}_3, \]
\[ \mathbf{A}(\mathbf{v}_1 - \mathbf{v}_3) = -\mathbf{v}_1 - 2 \mathbf{v}_2 + \mathbf{v}_3 + 2 \mathbf{v}_1 - \mathbf{v}_3 = \mathbf{v}_1 - 2 \mathbf{v}_2. \]

Consider
\[ \alpha \mathbf{A}(\mathbf{v}_1 + 2 \mathbf{v}_2) + \beta \mathbf{A} \mathbf{v}_2 + \gamma \mathbf{A}(\mathbf{v}_1 - \mathbf{v}_3) = \alpha (-\mathbf{v}_1 - 4 \mathbf{v}_2 + 5 \mathbf{v}_3) + \beta (-2 \mathbf{v}_2 + 2 \mathbf{v}_3) + \gamma (\mathbf{v}_1 - 2 \mathbf{v}_2)^* = 0 \]

or
\[ (-\alpha + \gamma) \mathbf{v}_1 + (-4 \alpha - \beta - 2 \gamma) \mathbf{v}_2 + (5 \alpha + 2 \beta) \mathbf{v}_3 = 0. \]

By linear independence of \( \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \} \), we have
\[ -\alpha + \gamma = 0, \]
\[ -4 \alpha - \beta - 2 \gamma = 0, \]
\[ 5 \alpha + 2 \beta = 0. \]

Hence, \( \{ \mathbf{v}_1 + 2 \mathbf{v}_2, \mathbf{A} \mathbf{v}_2, \mathbf{A}(\mathbf{v}_1 - \mathbf{v}_3) \} \) is linearly independent. ANS: True

(g) Consider the characteristic equation of \( \mathbf{A} \) given by
\[ \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ -1 & -\lambda & 1 \\ 1 & -1 & -\lambda \end{bmatrix} = (2 - \lambda)(\lambda^2 + 1) = 0. \]

Roots: \( \lambda_1 = 2, \lambda_{2,3} = \pm i \).

Since the vector space is real, hence only \( \lambda_1 \) is an eigenvalue of \( \mathbf{A} \). Its corresponding eigenvector only spans an one dimensional vector space. Hence \( \mathbf{A} \) is not simple. ANS: False

(h) Given: \( \mathbf{A} \mathbf{v}_1 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3, \mathbf{A} \mathbf{v}_2 = -3 \mathbf{v}_2, \mathbf{A} \mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_3. \)

\[ \det(\mathbf{A}) = \begin{vmatrix} 1 & 0 & 1 \\ 1 & -3 & 0 \\ -1 & 0 & -1 \end{vmatrix} = 0 \Rightarrow \mathbf{A} \text{ is singular} \Rightarrow \mathbf{A} \mathbf{x} = 0 \text{ for some nonzero } \mathbf{x}. \]

In fact, \( \mathbf{A}(\mathbf{v}_1 + 2 \mathbf{v}_2 - 3 \mathbf{v}_3) = 0. \) ANS: True

(i) Given \( \mathbf{A} \) and \( \mathbf{B} \) are distinct simple linear transformations on the vector space \( \mathcal{V} \), can \( \mathbf{C} = \mathbf{A} \mathbf{B}^2 \) be simple?

Let \( \mathbf{A} = \mathbf{I} \) (identity), \( \mathbf{B} = 2 \mathbf{I} \) (both \( \mathbf{A} \) and \( \mathbf{B} \) are simple). Then \( \mathbf{C} = 2 \mathbf{I} \) is simple. ANS: False

Prob. 2 (a)
\[ \mathbf{A} \mathbf{v}_1 = [1 0 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = [1 0 0] = \mathbf{v}_1; \]
\[ \mathbf{A} \mathbf{v}_2 = [1 1 0] \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = [1 -1 0] = 2 \mathbf{v}_1 - \mathbf{v}_2; \]
\[ \mathbf{A} \mathbf{v}_3 = [1 1 0] \]
\[
A = \begin{bmatrix}
0 & 1 & -1 \\
1 & 0 & 0 \\
0 & -1 & 0 \\
1 & 1 & 1
\end{bmatrix}
= \begin{bmatrix}
-1 & -2 & -1 \\
2v_1 & -3v_2 & v_3
\end{bmatrix}.
\]

Then, the matrix representation of \( A \) with respect to basis \( B \) is
\[
[A]_B = \begin{bmatrix}
1 & 2 & 2 \\
0 & -1 & -3 \\
0 & 0 & 1
\end{bmatrix}.
\]

(b) By inspection, the eigenvalues of \( A \) are: \( \lambda_1 = -1, \lambda_2, \lambda_3 = 1 \) (algebraic multiplicity = 2). The geometric multiplicity = 1, since the eigenvectors of \( A \) corresponding to \( \lambda_2, \lambda_3 = 1 \) are nonzero scalar multiples of \( v_1 \). Hence, \( A \) is not a simple linear transformation.

(c) Since \( A \) has no zero eigenvalues, thus \( A \) is nonsingular \( \rightarrow \) null space of \( A = \{0\} \), and the range of \( A \) is the whole space \( V \). Hence the nullity of \( A \) is zero, and the rank of \( A \) is 3.

(d) Let \( x(k) = \sum_{i=1}^{3} x_i v_i \). Hence the matrix representation of \( x(k+1) = Ax(k) \) is
\[
[x(k+1)]_B = [A]_B [x(k)]_B = \begin{bmatrix}
x_1(k) \\
x_2(k) \\
x_3(k)
\end{bmatrix}.
\]
The equilibrium state \( x_{eq} \) of this system is given by:
\[
[x_{eq}]_B = [A]_B [x_{eq}]_B \quad \text{or} \quad [x_{eq}]_B = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}, \quad [x_{eq}]_B = \begin{bmatrix}
a \\
0 \\
0
\end{bmatrix}, \quad a \text{ is any real number},
\]
or \( x_{eq} = Av_i = [a \ 0 \ 0] \).

Prob.3 (a) The equivalent single second-order differential equation is:
\[
d^2x_1/dt^2 - 2t^{-2}x_1 = u(t).
\]
We seek solutions to the above equation with \( u(t) = 0 \) in the form \( x_1(t) = t^k \). By direct substitution, we have
\[
k(k-1)t^{k-2} - 2t^{k-2} = 0 \Rightarrow k^2 - k - 2 = 0. \quad \text{Roots: } k_1 = -1, \ k_2 = 2.
\]
This gives two solutions to the homogeneous equation:
\[
x_1^{(1)}(t) = t^{-1}, \quad x_1^{(2)}(t) = t^2.
\]
To check their linear independence, we compute the Wronskian:
\[
W(x_1^{(1)}(t), x_1^{(2)}(t)) = \begin{vmatrix}
x_1^{(1)}(t) & x_1^{(2)}(t) \\
x_1^{(1)}(t) & x_1^{(2)}(t) + t^2\end{vmatrix} = t^{-1}2t - t^2(-t^{-2}) = 3.
\]
Thus the general solution to the homogeneous equation is
\[
x_1(t) = c_1 t^{-1} + c_2 t^2, \quad c_1, c_2 \text{ are arbitrary real numbers}.
\]
(b) Given: \( u(t) = t \). Find the general solution to the nonhomogeneous equation. By the method of variation of parameters, here, we seek a particular solution in the form:

\[
z(t) = v_1(t)x^{(1)}(t) + v_2(t)x^{(2)}(t) = t^{-1}v_1(t) + t^2v_2(t).
\]

We compute:

\[
dz(t)/dt = t^{-1}v_1(t) - t^{-2}v_1(t) + 2tv_2(t) + t^2v_2(t)
\]

and set

\[
t^{-1}v_1(t) + t^2v_2(t) = 0 \quad \Rightarrow \quad v_1(t) = -t^3v_2(t).
\]

Thus,

\[
d^2z(t)/dt^2 = 2t^{-3}v_1(t) - t^{-2}v_1(t) + 2v_2(t) + 2tv_2(t).
\]

Substituting the above expressions into the nonhomogeneous differential equation gives:

\[
2t^{-3}v_1 - t^{-2}v_1 + 2v_2 + 2tv_2 - 2t^{-2}(t^{-1}v_1 + t^2v_2) = -t.
\]

or

\[
-t^{-2}v_1 + 2tv_2 = -t \quad \Rightarrow \quad \dot{v}_1 - 2t^3v_2 = t.
\]

From (1) and (2), we have

\[
3v_2(t) = -1 \quad \Rightarrow \quad v_2(t) = -t/3 \quad \Rightarrow \quad \dot{v}_1(t) = t^3/3 \quad \Rightarrow \quad v_1(t) = t^4/12.
\]

Thus, we have a particular solution

\[
z(t) = t^{-1}v_1(t) + t^2v_2(t) = \frac{1}{12}t^3 - \frac{1}{3}t^3 = -t^3/4.
\]

So the general solution is given by

\[
x(t) = x(t) + z(t) = c_1t^{-1} + c_2t^2 - t^3/4
\]