Consider the following $N$-dimensional linear time-invariant system:

$$\frac{dx}{dt} = Ax + Bu, \quad y = Cx,$$

where $A$, $B$, and $C$ are specified $N \times N$, $N \times K$, and $M \times N$ constant matrices respectively. Suppose that the system state $x(t)$ can be measured at any time $t$. Then we can construct a linear feedback control of the form:

$$u(t) = Fx(t) + v(t),$$

where $F$ is a constant $K \times N$ feedback gain matrix, and $v$ is a new input to the system. The corresponding feedback control system

$$\frac{dx}{dt} = (A + BF)x + Bv(t)$$

can be made to have certain desired dynamic behavior by choosing a suitable $F$. In particular, if system (1) is completely controllable, then we can assign the spectrum of $(A + BF)$ or the poles of the system transfer function by choosing an appropriate $F$.

Now, suppose that not all components of $x(t)$ can be measured directly. We wish to use feedback control (2) with $x(t)$ replaced by $\hat{x}(t) = Rx(t) + Sz(t)$, where $Rx(t)$ corresponds to that part of $x(t)$ which can be measured directly. For example, if the first $P$ components of $x(t)$ can be measured directly, then $R$ is a $N \times N$ matrix of the form

$$R = \begin{bmatrix} I_P & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

where $I_p$ is a $P \times P$ identity matrix. The vector $Sz(t)$ corresponds to those components of $x(t)$ which cannot be measured directly, and they are estimated by an asymptotic observer of the form:

$$\frac{dz}{dt} = Hz + Gy + Qu,$$

where $z$ is related to $x$ by a linear transformation $T$, i.e. $z = Tx; H, G$, and $Q$ satisfy the following equations (see Lecture Notes 12):

$$TA - HT = GC, \quad TB = Q.$$  \hfill (6)

Since

$$\dot{x} = Rx + STx = (R + ST)x,$$

and $\dot{x}$ corresponds to an estimate of the state $x$, therefore

$$R + ST = I.$$  \hfill (7)
A block diagram of the feedback control system is shown below:

Now, the main question is: what are the properties of the composite feedback control and observer system, in particular, where are the eigenvalues (poles) of the composite system? To answer the latter question, we first rewrite the equation of the composite system in the following form:

\[
\frac{d}{dt} \begin{bmatrix} x \\ z \end{bmatrix} = A_c \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B \\ TB \end{bmatrix} v, \tag{8}
\]

where

\[
A_c = \begin{bmatrix} A + BFR & BFS \\ GC + TBFR & H + TBFS \end{bmatrix}. \tag{9}
\]

We shall prove the following theorem which provides the answer to the foregoing question.

**Theorem.** Assuming that (6) and (7) hold, then the eigenvalues of \( A_c \) in the composite system (8) are those of \((A + BF)\) and \(H\).

**Proof.** Let \( \lambda_i \) be an eigenvalue of \( A_c \) and

\[
\begin{bmatrix} x^{(i)} \\ z^{(i)} \end{bmatrix}
\]

be an eigenvector corresponding to eigenvalue \( \lambda_i \). Thus, \( A_c w^{(i)} = \lambda_i w^{(i)} \) implies

\[
(A + BFR)x^{(i)} + BFSz^{(i)} = \lambda_i x^{(i)} \tag{10}
\]

\[
(GC + TBFR)x^{(i)} + (H + TBFS)z^{(i)} = \lambda_i z^{(i)} \tag{11}
\]

Premultiplying (10) by \( T \) gives

\[
(TA + TBFR)x^{(i)} + TBFSz^{(i)} = \lambda_i Tx^{(i)}. \tag{12}
\]
Now, subtracting (11) from (12) leads to
\[(TA - GC)x^{(i)} - Hz^{(i)} = \lambda_i(Tx^{(i)} - z^{(i)}),\]  
(13)

which, in view of (6), reduces to
\[H(Tx^{(i)} - z^{(i)}) = \lambda_i(Tx^{(i)} - z^{(i)}).\]  
(14)

Evidently, if \(Tx^{(i)} \neq z^{(i)}\), then \((Tx^{(i)} - z^{(i)})\) is an eigenvector of \(H\) corresponding to eigenvalue \(\lambda_i\). In fact, all the eigenvalues of \(H\) (including multiplicities) are eigenvalues of \(A_c\).

Now, (14) is also satisfied if \(Tx^{(i)} = z^{(i)}\). Under this condition, (10) becomes
\[\begin{bmatrix} A + BF \end{bmatrix}(R + ST)x^{(i)} = \lambda_i x^{(i)}\]  
(15)

From the assumption that \(R + ST = I\), (15) becomes
\[(A + BF)x^{(i)} = \lambda_i x^{(i)}\]  
(16)

which implies that \(\lambda_i\) is an eigenvalue of \((A + BF)\). By reversing the foregoing process, we conclude that every eigenvalue of \((A + BF)\) or \(H\) is also an eigenvalue of \(A_c\). This completes the proof. \[\|

The foregoing theorem implies that the feedback control system can be designed by assuming that the state \(x(t)\) is completely known (i.e. Choose a \(F\) such that \((A + BF)\) has the desired spectrum), and then construct an observer to obtain an estimate of the state \(x(t)\). These two tasks can be performed independently. This important result is known as the separation principle. This result can be extended to the case where the output is corrupted by random noise.