Lecture Notes 2

Mathematical Models in the Form of Ordinary Difference and Differential Equations

1. Difference Equations:

Simple Examples:

a. \( x(k + 1) = \alpha x(k) \), \( \alpha \) is a given real number, \( k = 0, 1, 2, \ldots \) \hspace{1cm} (1)

Given \( x(0) = x_o \), the solution to this equation can be obtained recursively: i.e.

\[
\begin{align*}
  x(1) &= \alpha x(0) = \alpha x_o, \\
  x(2) &= \alpha x(1) = \alpha^2 x_o, \\
  x(3) &= \alpha x(2) = \alpha^3 x_o, \ldots, \\
  x(k) &= \alpha^k x_o, \quad k = 0, 1, 2, \ldots
\end{align*}
\]

Remarks:

R.1 If \( |\alpha| < 1 \), then \( |x(k)| \to 0 \) as \( k \to \infty \).

(The zero solution \( (x(k) = 0 \) for all \( k \)) is said to be asymptotically stable.)

If \( |\alpha| > 1 \), then \( |x(k)| \to \infty \) as \( k \to \infty \).

(The zero solution is said to be unstable.)

R.2 Given \( x(i) \) at any \( i \), the solution \( x(k) \) for \( k > i \) is uniquely determined. Hence \( x(i) \) corresponds to the state of the system at \( i \).

Application to Chain letter: Let \( x(k) \) denote the number of letters one received in the \( k \)th generation, \( \alpha \) the number of letters one sent for each letter received, i.e. \( x(0) \) denotes the number of letters one received, \( x(1) \) the number of letters sent out, \( x(2) \) the number of letters written by those contacted, and so on. Then, equation (1) describes the number of letters one received in the \( (k + 1) \)th generation. For example, let \( x(0) = 1 \), and \( \alpha = 10 \). Then, \( x(k) = 10^k \). So, if one enclosed \$1 in each letter, then after six generations \( x(6) = 10^6 \) which corresponds to one million dollars.

b. \( x(k + 1)^2 = x(k) + x(k - 1) \), \( k = 0, 1, 2, \ldots \) \hspace{1cm} (2)

Here, at \( k = 0 \), we need to know both \( x(0) \) and \( x(-1) \) in order to find \( x(1) \). Moreover, if we restrict the solution \( x(k) \) to be real numbers, then \( x(0) \) and \( x(-1) \) must be real numbers such that \( x(0) + x(-1) \geq 0 \).

Note: There are two possible values for \( x(1) \):

\[
  x(1) = \sqrt{x(0) + x(-1)} \quad \text{and} \quad x(1) = -\sqrt{x(0) + x(-1)}.
\]

Thus, the solution is generally nonunique.

Now,

\[
x(2)^2 = x(1) + x(0) = \sqrt{x(0) + x(-1)} + x(0) \quad \text{(taking the positive root for } x(1))
\]

Again, we have two possible values for \( x(2) \):
\[ x(2) = \pm \sqrt{x(0) + x(-1) + x(0)}. \]

**Question:** Can we determine real solutions for all \( k > 0 \)?

It is evident that the existence of a real solution for all \( k > 0 \) depends on the choice of \( x(0) \) and \( x(-1) \).

Let \( x(0) = 0, x(-1) = 1 \). Then, the positive branch of the solution is

\[
x(1) = \sqrt{x(0) + x(-1)} = 1, \quad x(2) = \sqrt{x(1) + x(0)} = 1,
\]

\[
x(3) = \sqrt{x(2) + x(1)} = \sqrt{2}, \quad x(4) = \sqrt{x(3) + x(2)} = \sqrt{\sqrt{2} + 1}, \ldots
\]

Clearly, the solution \( x(k) \) can be determined for all \( k > 0 \).

Now, let \( x(0) = -1, x(-1) = 1 \). Then, we have

\[
x(1) = \sqrt{x(0) + x(-1)} = 0, \quad x(2) = \sqrt{x(1) + x(0)} = \sqrt{-1} \text{ (not real)}.
\]

Thus, we cannot continue the solution for \( k > 1 \).

**Exercise:** Explore the existence of real solutions for all initial states in the plane \( R^2 \).

**Remarks:**

**R.3** Here, at any \( k \), we need both \( x(k) \) and \( x(k-1) \) in order to determine \( x(k+1) \). Thus, the state of the system at the \( k \)th step is specified by the pair \((x(k), x(k-1))\), or a point (a “vector”) in the real plane \( R^2 \).

**R.4** Equation (2) is called an *implicit* difference equation, since we must *solve* for \( x(k+1) \) in terms of \( x(k) \) and \( x(k-1) \).

### 1.1 General Scalar Ordinary Difference Equation in Explicit Form:

\[
x(k+n) = h(x(k+n-1), x(k+n-2), \ldots, x(k), k), \quad k = k_o, k_o + 1, \ldots, \quad (3)
\]

where \( h \) is a given real-valued function of its arguments, \( k_o \) is an integer, and \( n \) is a given positive integer specifying the order of the difference equation.

Here, given \( x(k_o + n - 1), x(k_o + n - 2), \ldots, x(k_o) \) (the initial conditions), \( x(k_o + n) \) is uniquely determined. Thus, \( x(k+n) \) can be determined recursively for all \( k > k_o \), implying the existence of a unique solution.

Equation (3) can be rewritten as a system of *first-order* difference equations by defining:

\[
x_1(k) = x(k),
\]

\[
x_2(k) = x(k+1),
\]

\[
x_3(k) = x(k+2),
\]

\[ \vdots \]

\[
x_{n-1}(k) = x(k+n-2),
\]

\[
x_n(k) = x(k+n-1).
\]
Thus,

\[ x_1(k + 1) = x(k + 1) = x_2(k), \]
\[ x_2(k + 1) = x(k + 2) = x_3(k), \]
\[ x_3(k + 1) = x(k + 3) = x_4(k), \]
\[ \vdots \]
\[ x_{n-1}(k + 1) = x(k + n - 1) = x_n(k), \]
\[ x_n(k + 1) = x(k + n) = h(x_1(k), \ldots, x_n(k)). \]

Let \( x(k) \) denote the ordered \( n \)-tuple \((x_1(k), \ldots, x_n(k))\). The above equations can be rewritten as a first-order difference equation:

\[ x(k + 1) = f(x(k), k), \] (5)

where \( f = (f_1, \ldots, f_n) \) defined by the right-hand-sides of (4). If the \( x_i(k) \)'s are real numbers, then the solution points can be regarded as a point in the \( n \)-dimensional real coordinate space \( \mathbb{R}^n \). Given \( x(k_0) \) (the state of the system (5) at \( k_0 \)), the solution \( x(k) \) is uniquely determined for all \( k > k_0 \), i.e.

\[ x(k_0 + 1) = f(x(k_0), k_0), \]
\[ x(k_0 + 2) = f(x(k_0 + 1), k_0 + 1) = f(f(x(k_0), k_0), k_0 + 1), \]
\[ x(k_0 + 3) = f(x(k_0 + 2), k_0 + 2) = f(f(x(k_0), k_0), k_0 + 1), k_0 + 2), \]
\[ \vdots \]

We note that the right-hand-side of (5) takes a point \( x(k) \) in \( \mathbb{R}^n \) into a new point \( x(k+1) \) in \( \mathbb{R}^n \) defined by \( f(x(k), k) \). Therefore, we also call such a system, a point-mapping system.

In the case of a scalar difference equation of the form:

\[ x(k + 1) = f(x(k)), \quad k = 0, 1, 2, \ldots, \] (6)

with initial condition \( x(0) = x_0 \). We can solve this equation graphically as shown in Fig.1. First, we plot the graph of \( f \), and draw a diagonal line. Then the successive solution points can be obtained by starting with \( x_0 \) and reflecting \( f(x_0) \) about the diagonal line.

We note that in this example, there are points \( x^* \) such that \( x^* = f(x^*) \). If we start with the initial condition \( x(0) = x^* \), then \( x(k) = x^* \) for all \( k > 0 \). This point is called an equilibrium point of equation (6), or a fixed point of the mapping \( f \).

This notion of equilibrium point can be extended to the general case (5). Here, an equilibrium point or (equilibrium state) is a point \( x^* \) such that \( x^* = f(x^*, k) \) for all \( k \).

The set of all equilibrium states of (5) can be determined by solving the algebraic equation:

\[ x = f(x, k) \] for all \( k \). (7)
Equation (7) corresponds to a set of nonlinear algebraic equations. This equation may have many solutions or no solutions at all.

**Example:** Let $\Sigma = [0, 1]$. The scalar equation (logistic equation for population growth)

$$x(k + 1) = f(x(k), \mu) \overset{\text{def}}{=} \mu x(k)(1 - x(k))$$

defined on $\Sigma$, where $\mu$ is a real number. In order that $x(k)$ belongs to $\Sigma$ for all $k$, the parameter $\mu$ must satisfy $0 \leq \mu \leq 4$. This equation has two equilibrium states: $x = 0$ and $x = 1 - \mu^{-1}$. The solution to this equation may have complex behavior depending on the value of $\mu$.

**Example:** The equation $x^2 = a$ can be rewritten as

$$x = \frac{1}{2} \left( x + \frac{a}{x} \right). \quad (8)$$

Therefore, $\sqrt{a}$ is an equilibrium point of the following difference equation:

$$x(k + 1) = \frac{1}{2} \left( x(k) + \frac{a}{x(k)} \right) \overset{\text{def}}{=} f(x(k), a) \quad (9)$$

Evidently, a root of (8) is an equilibrium state of (9), or a fixed point of the mapping $f(\cdot, a)$. This equation can be used to calculate $\sqrt{a}$ iteratively. For example, let $a = 2$ and $x(0) = 3$. Then

$$x(1) = 1.83333, \quad x(2) = 1.46212, \quad x(3) = 1.41499, \quad x(4) = 1.41421, \ldots$$
**Example:** $f$ is a mapping defined on the square $\Sigma = [0, 1] \times [0, 1]$ into itself (i.e. $f$ maps points in $\Sigma$ into $\Sigma$) given by

$$f(x_1, x_2) = \begin{cases} \begin{bmatrix} 2x_1 \\ x_2/2 \end{bmatrix}, & \text{if } 0 \leq x_1 < 1/2; \\ \begin{bmatrix} 2x_1 - 1 \\ (x_2 + 1)/2 \end{bmatrix}, & \text{if } 1/2 \leq x_1 < 1. \end{cases} \quad (1)$$

This mapping (called the baker’s mapping derived from the baker’s manipulation of dough in making bread) preserves area. The action of $f$ on $\Sigma$ is shown in Fig.2

![Figure 2: Baker’s mapping.](image)

Note that this equation has nonzero equilibrium state $(1, 1)$ only. The baker’s mapping on $\Sigma = \text{unit cube in } \mathbb{R}^3$ is shown in the following figure. This mapping preserves volume, and it corresponds to the dough manipulation steps in making the Chinese “1000-layer cake”.

![Figure 3: Three-dimensional baker’s mapping.](image)

**Exercise:** Write down an explicit expression for the baker’s mapping defined on the unit cube.

**1.2 Linear Ordinary Difference Equations:** A scalar ordinary difference equation is said to be linear, if it has the form:

$$a_n(k)x(k+n) + a_{n-1}(k)x(k+n-1) + \ldots + a_1(k)x(k+1) + a_0(k)x(k) = u(k), \quad (11)$$
where \( a_i(k), i = 0, 1, 2, \ldots, n, u(k) \) are given functions of \( k \). The variable \( u(k) \) is usually called the input, forcing function, or control.

If \( a_i(k), i = 0, 1, 2, \ldots, n \) are specified constants, then the equation is said to have constant coefficients (or time-invariant, if \( k \) corresponds to time instants). Otherwise, the equation is said to have variable coefficients (or time-varying coefficients, if \( k \) corresponds to time instants).

If \( u(k) \equiv 0 \) for all \( k \), then equation (11) is said to be homogeneous.

In what follows, we shall make a few important observations pertaining to the solutions of (11).

**Theorem 2.1.** Let \( \bar{x} = \bar{x}(k) \) be a particular solution to the linear difference equation (11). Then all the solutions to (11) has the form \( x(k) = \bar{x}(k) + z(k) \), where \( z = z(k) \) is a solution of the homogeneous equation corresponding to (11) (i.e. (11) with \( u(k) \equiv 0 \) for all \( k \)).

**Proof:** First, we show that if \( x(k) \) and \( \bar{x}(k) \) are both solutions to the nonhomogeneous equation (11), then the difference \( z(k) = x(k) - \bar{x}(k) \) is a solution to the homogeneous equation. To show this, since \( x(k) \) and \( \bar{x}(k) \) are both solutions to (11), we have

\[
a_n(k)x(k + n) + \ldots + a_0(k)x(k) = a_n(k)\bar{x}(k + n) + \ldots + a_0(k)\bar{x}(k).
\]

Rearranging terms gives

\[
a_n(k)[x(k + n) - \bar{x}(k + n)] + \ldots + a_0(k)[x(k) - \bar{x}(k)] = 0.
\]

Therefore, the difference \( z(k) = x(k) - \bar{x}(k) \) satisfies the homogeneous equation.

Now, we show that any solution of the homogeneous equation added to a particular solution of the nonhomogeneous equation (11) is also a solution of the nonhomogeneous equation (11). To show this, let \( \bar{x}(k) \) and \( z(k) \) be solutions to the nonhomogeneous and homogeneous equations respectively. Let \( x(k) = \bar{x}(k) + z(k) \). Then,

\[
a_n(k)[\bar{x}(k + n) + z(n + k)] + \ldots + a_0(k)[\bar{x}(k) + z(k)]
\]

\[
= a_n(k)\bar{x}(k + n) + \ldots + a_0(k)\bar{x}(k) + a_n(k)z(k + n) + \ldots + a_0(k)z(k)
\]

\[
= u(k) + 0 = u(k).
\]

Therefore, \( x(k) = \bar{x}(k) + z(k) \) is a solution of the nonhomogeneous equation (11).

**Example:** Consider the equation \( x(k + 1) = \mu x(k) + a \).

The homogeneous equation \( x(k + 1) = \mu x(k) \) has solutions of the form:

\[
z(k) = C\mu^k,
\]

where \( C \) is an arbitrary constant.

A solution to the nonhomogeneous equation is

\[
\bar{x}(k) = a/(1 - \mu) \text{ if } \mu \neq 1,
\]

which is an equilibrium state of the equation. It follows from Theorem 2.1 that
\[ x(k) = C \mu^k + a/(1 - \mu) \]

is a solution of the nonhomogeneous equation for \( \mu \neq 1 \). For \( \mu = 1 \), the solutions to the homogeneous equation are constants, and \( \bar{x}(k) = ak \) is a particular solution of the nonhomogeneous equation. Thus, the general solution has the form: \( x(k) = C + ak, k = 0, 1, 2, \ldots \).

**Theorem 2.2.** If \( z_i(k), i = 1, \ldots, m, \) are all solutions to the homogeneous equation corresponding to (11), then any linear combination of these \( m \) solutions of the form

\[ w(k) = \sum_{i=1}^{m} c_i z_i(k), \]

where \( c_i \)'s are arbitrary constants, is also a solution of the homogeneous equation.

*Proof:* We write

\[
\begin{align*}
  a_n(k) & \left\{ \sum_{i=1}^{m} c_i z_i(k + n) \right\} + \ldots + a_0(k) \left\{ \sum_{i=1}^{m} c_i z_i(k) \right\} \\
  & = \sum_{i=1}^{m} c_i \{ a_n(k) z_i(k + n) + \ldots + a_0(k) z_i(k) \} .
\end{align*}
\]

Since each \( \{ \cdots \} \) term in the last expression is zero, hence we have

\[ a_n(k) w(k + n) + \ldots + a_0(k) w(k) = 0. \]

Now, we focus our attention on the homogeneous difference equation

\[ a_n(k) x(k + n) + \ldots + a_0(k) x(k) = 0. \quad (12) \]

Let \( \bar{z}_i = \bar{z}_i(k), i = 1, \ldots, n \) denote the solution to (12) with initial conditions:

\[
\begin{align*}
  \bar{z}_1(0) & = 1, \quad \bar{z}_1(1) = 0, \ldots, \bar{z}_1(n - 1) = 0, \\
  \bar{z}_2(0) & = 0, \quad \bar{z}_2(1) = 1, \quad \bar{z}_2(2) = 0, \ldots, \bar{z}_2(n - 1) = 0, \\
  & \vdots \\
  \bar{z}_n(0) & = 0, \quad \bar{z}_n(1) = 0, \quad \bar{z}_n(2) = 0, \ldots, \bar{z}_n(n - 1) = 1,
\end{align*}
\]

The set of solutions \( \bar{z}_1(k), \ldots, \bar{z}_n(k) \) is called a *fundamental set* of solutions to (12).

**Theorem 2.3.** If \( z = z(k) \) is any solution to the homogeneous equation (12), then \( z(k) \) can be expressed in terms of the fundamental solutions \( \bar{z}_i \) in the form:

\[ z(k) = \sum_{i=1}^{n} c_i \bar{z}_i(k) \]

for some constant \( c_1, \ldots, c_n \).
Proof: Let \( z = z(k) \) be an arbitrary solution to (12). Corresponding to its initial values define:

\[ c_i = z(i - 1), i = 1, \ldots, n. \]

Now, consider the special solution \( x(k) \) defined by

\[ x(k) = c_1 \bar{z}_1(k) + \ldots + c_n \bar{z}_n(k). \]

This solution has the same \( n \) initial conditions as the original solution \( z(k) \). Therefore it follows from the existence and uniqueness of solutions that \( x(k) = z(k) \).

Example: Consider the linear homogeneous difference equation:

\[ x(k + 2) - 2x(k + 1) + x(k) = 0. \]

To find the fundamental solution \( \bar{z}_1(k) \), we consider the initial conditions \( x(0) = 1, x(1) = 0 \). Thus,

\[
\begin{align*}
x(2) &= 2x(1) - x(0) = -1, \\
x(3) &= 2x(2) - x(1) = -2 - 0 = -2, \\
x(4) &= 2x(3) - x(2) = -4 + 1 = -3, \\
x(5) &= 2x(4) - x(3) = -6 + 2 = -4, \\
x(6) &= 2x(5) - x(4) = -8 + 3 = -5, \\
&\vdots \\
\end{align*}
\]

Thus, \( \bar{z}_1(k) = 1 - k \).

To find the fundamental solution \( \bar{z}_2(k) \), we consider the initial conditions \( x(0) = 0, x(1) = 1 \). Thus,

\[
\begin{align*}
x(2) &= 2x(1) - x(0) = 2, \\
x(3) &= 2x(2) - x(1) = 4 - 1 = 3, \\
x(4) &= 2x(3) - x(2) = 6 - 2 = 4, \\
&\vdots \\
\end{align*}
\]

Thus, \( \bar{z}_2(k) = k \).

An arbitrary solution has the form:

\[ x(k) = c_1 \bar{z}_1(k) + c_2 \bar{z}_2(k) = c_1(1 - k) + c_2k = c_1 + (c_1 - c_2)k. \]

Since \( c_1 \) and \( c_2 \) are arbitrary, therefore \( x(k) \) can be rewritten as

\[ x(k) = c + dk, \]
where \( c \) and \( d \) are arbitrary constants.

**Definition:** Given a set of functions \( z_1 = z_1(k), \ldots, z_m = z_m(k) \), defined for integers \( k = 0, 1, 2, \ldots \). This set is said to be linearly independent if and only if

\[
\sum_{i=1}^{m} c_i z_i(k) = 0 \tag{\text{(*)}}
\]

implies that \( c_1 = c_2 = \ldots = c_m = 0 \).

If there does not exist \( c \)'s, all zero, such that (\text{(*)}) is satisfied, then the set of functions \( z_1 = z_1(k), \ldots, z_m = z_m(k) \) is said to be linearly dependent.

**Example:** The set \( \{ z_1(k) = 1, z_2(k) = 2^k, z_3(k) = 3^k, k = 0, 1, 2, \ldots \} \) is linearly independent, since

\[
c_1 z_1(k) + c_2 z_2(k) + c_3 z_3(k) = c_1 + c_2 2^k + c_3 3^k = 0 \text{ for } k = 0, 1, 2, \ldots
\]

implies \( c_1, c_2, c_3 = 0 \). Note that the above equation means

- for \( k = 0 \) : \( c_1 + c_2 + c_3 = 0 \),
- for \( k = 1 \) : \( c_1 + 2c_2 + 3c_3 = 0 \),
- for \( k = 2 \) : \( c_1 + 4c_2 + 9c_3 = 0 \), etc.

These three equations are sufficient for determining \( c_1, c_2 \) and \( c_3 \). The only solution is \( c_i = 0, i = 1, 2, 3 \).

**Example:** The set \( \{ z_1(k) = 1, z_2(k) = k, z_3(k) = 5k, k = 0, 1, 2, \ldots \} \) is linearly dependent, since there exist \( c_1, c_2 \) and \( c_3 \), not all zero, such that

\[
c_1 z_1(k) + c_2 z_2(k) + c_3 z_3(k) = c_1 + c_2 k + 5c_3 k = 0 \text{ for } k = 0, 1, 2, \ldots
\]

i.e.

- for \( k = 0 \) : \( c_1 = 0 \),
- for \( k = 1 \) : \( c_1 + c_2 + 5c_3 = 0 \Leftrightarrow c_2 + 5c_3 = 0 \),
- for \( k = 2 \) : \( c_1 + 2c_2 + 10c_3 = 0 \Leftrightarrow c_2 + 5c_4 = 0 \),
- for \( k = 3 \) : \( c_1 + 3c_2 + 15c_3 = 0 \Leftrightarrow c_2 + 5c_3 = 0 \),

\vdots

We choose \( c_2 = -5c_3 \), e.g. \( c_2 = 1 \) and \( c_3 = -1/5 \).

The result is also obvious from the fact that \( z_3 = 5z_2 \).

With the notion of linear independence of a set of functions, we can extend Theorem 2.3 as follows:

**Theorem 2.4.** Suppose \( \{ z_1(k), \ldots, z_n(k) \} \) is a linearly independent set of solutions to the homogeneous difference equation (12). Then any solution \( x(k) \) to (12) can be expressed in the form:
\[ x(k) = \sum_{i=1}^{n} c_i z_i(k) \]

for some constants \( c_1, \ldots, c_n \).

**Interpretation:** If we can find a set of \( n \) linearly independent solutions to the homogeneous difference equation, then we can obtain its general solution as a linear combination of these linearly independent solutions.

For the nonhomogeneous linear difference equation (11), we have the following result:

**Theorem 2.5.** If \( \bar{x} = \bar{x}(k) \) is a particular solution to the nonhomogeneous linear difference equation (11) and \( z_1(k), \ldots, z_n(k) \) are linearly independent solutions to the corresponding homogeneous equation, then the general solution to (11) has the form:

\[ x(k) = \bar{x}(k) + \sum_{i=1}^{n} c_i z_i(k). \]

**Note:** If a different particular solution is used, it would simply change the values of the coefficients \( c_i \).