Prob. 1 (a) Write down a first-order difference equation of the form \( x(k+1) = f(x(k)) \) to find the positive root of \( x^3 = \alpha \), where \( \alpha \) is a positive number. The positive root should be a fixed point of \( f \).

(b) Find the positive cube root of \( \alpha = 3 \) iteratively by solving your difference equation starting with some initial guess \( x(0) \). Does the solution converge to the desired root for any positive \( x(0) \)? Explain.

(c) Can your difference equation be used to find the negative cubic root of a negative number \( \alpha \)? Justify your answer.

Prob. 2 Assuming \( y(0) = 0, \ y(1) = y(2) = 1 \), find, by direct numerical recursion, the values of \( y(k) \), \( k = 3,4,5 \), satisfying the difference equation:

\[
y(k+3) - y(k+2) + [y(k+1) - y(k)]^2 = 1.
\]

Prob. 3 Numerical solution of differential equations: Differential equations are often solved numerically by a discrete forward recursion method. Consider the scalar differential equation:

\[
dx/dt = f(x).
\]

To solve this equation numerically, one considers the sequence of discrete points 0, s, 2s, 3s, ..., where \( s \) is a positive "step length."

The simplest solution technique is the Euler's method, which calculates a sequence of values \( x(k) \) according to the recursion:

\[
x(k+1) = x(k) + sf(x(k)), \quad (*)
\]

where \( s \) is the step size, and we have approximated \( dx/dt \) at \( t = ks \) by the forward difference: \( [x(k+1) - x(k)]/s \).

(a) Assuming \( f(x) = \alpha x \), for what range of values of the constant \( \alpha \) does the solution of \((*)\) converge to zero as \( t \to \infty \)?

(b) For a fixed \( \alpha < 0 \), what is the largest step length that guarantees convergence in Euler's method?

Prob. 4 Find the solutions to the following difference equations, for \( k = 0,1,2, \ldots \)

(a) \( x(k+2) - 5x(k+1) + 6x(k) = 0, \ x(0) = x(1) = 1 \);

(b) \( x(k+2) - 3x(k+1) + 2x(k) = 1, \ x(0) = 1, \ x(1) = -1 \).

Prob. 5: Consider the one-dimensional point-mapping system defined on \( \Sigma = [0,1] \) described by the following difference equation:

\[
x(k+1) = f(\mu,x(k)), \quad f(\mu,x) = \mu x(1-x),
\]

where \( \mu \) is a real parameter.

(a) Determine the range of parameter \( \mu \) such that \( f \) maps \( \Sigma \) into \( \Sigma \). (i.e. If \( x \) is a point in \( \Sigma \), then \( f(\mu,x) \) is also a point in \( \Sigma \).)

(b) Determine the equilibrium states of the system as a function of \( \mu \).

(c) Determine the behavior of the solution sequence for each of the following values of parameter \( \mu \): 1/2, 3/2, 3.75 and 3.85. Choose a nonequilibrium initial state at \( k = 0 \), and compute the subsequent states for sufficiently large \( k \) so you can observe the solution behavior. You may wish to use a PC to do this. Plot your results as functions of \( k \). Summarize your observations.